NEMETH CODE

of

BRAILLE MATHEMATICS

REVISED, 1956

By ABRAHAM NEMETH

Prepared for the
Sub-Committee on Mathematics of the Joint Uniform Braille Committee
Representing the
American Association of Workers for the Blind
and the
American Association of Instructors of the Blind

Printed by the
AMERICAN PRINTING HOUSE FOR THE BLIND
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1956

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INTRODUCTION

Mathematical expressions are represented in ink print by the use of a large number of arbitrary symbols. Among these symbols we find the digits, the small and capitalized letters of the Roman and Greek alphabets, the letters of other alphabets, script letters, letters in boldface type, letters written in Gothic style, signs of operation, signs of comparison, symbols of grouping, and other symbols serving the miscellaneous requirements of mathematical expression. Mathematical significance is conveyed not only by these symbols separately, but by their arrangement on levels above or below the level of the base line of writing.

In Braille, it is necessary to provide for the representation of all of these symbols, and also to provide a means of indicating their [IV] arrangement at the various levels of writing. It is, of course, impossible to represent each ink-print symbol by a distinct single Braille character, for there are at our disposal only 63 distinct Braille characters while there are many more ink-print symbols than this number. Furthermore, it is impractical to imitate the arrangement of these symbols in Braille by writing them at various levels above or below the base line of writing.

Accordingly, the system whose description follows is characterized by the use of Braille "indicators." These indicators serve to distinguish between small and capitalized letters, between the letters of one alphabet and those of another, between letters which appear in their usual form and those which appear in an altered form, between small and large ink-print symbols of the same kind, and between symbols which appear on the base line of writing and those which appear at levels above or below the base line of writing.

In evaluating any system of Braille notation, two questions should be kept in mind: First, are all necessary ink-print symbols and arrangements of ink-print symbols [V] representable in the system of Braille notation? Second, how would a transcript, made from the Braille text, compare with the original ink-print copy?

For the first question, it will be found that this proposed system makes possible the representation of mathematical expressions which either cannot be represented at all by the systems now in existence, or which require the use of Braille symbols which are arbitrary and have no ink-print equivalents. It is for this reason that blind students of the more advanced branches of mathematics should find this system adequate for their purposes.

For the second question, it will be found that an admirable likeness of the original ink-print copy will result from a transcription of the Braille text. For the transcriber, this means a uniform application of the rules of the system without the necessity of being constantly on the alert for possible exceptions.

One consequence of the close correspondence between the ink-print and the Braille text is that the latter is unencumbered by a constant use of [VI] symbols of grouping which do not occur in the ink print, thereby facilitating the work of the transcriber and giving the reader a more rapid and clearer grasp of the intended meaning. Another consequence of this correspondence is that the area of common ground between seeing teacher and blind student is increased. Thus, less time need be spent by the teacher in imparting the mechanics of the system, and more time may be devoted to the mathematical content which it conveys.

Appended to the description of the system is a selection of examples illustrating the application of the system to problems in arithmetic, algebra, geometry, trigonometry, calculus, and other branches of mathematics. The arrangement of the solutions is a matter of individual preference, and it should not be inferred that the arrangement here presented is the only one that may be followed.

ABRAHAM NEMETH

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NEMETH CODE OF BRAILLE MATHEMATICS

(Note: Bold-face numbers inclosed in brackets indicate the pagings of the Braille edition of the Nemeth Code.)

I

CHOICE BETWEEN USE OF MATHEMATICAL NOTATION OF STANDARD ENGLISH BRAILLE AND THAT OF THE NEMETH CODE

In the presentation of mathematical expressions in Braille, a choice must be made between the symbols and usages followed in Standard English Braille and those of the Nemeth Code of Braille Mathematics, based on whether the material being embossed is primarily literary or mathematical in nature.

1. **Literary Texts** — When writing mathematical expressions in non-mathematical or non-technical material, the rules and usages of Standard English Braille should be employed. However, when the material being embossed is mathematical or scientific in nature, the rules and usages of the Nemeth Code should be employed throughout.

In mathematical or scientific texts, words and mathematical expressions are separated from each other by leaving two blank **[2]** spaces, according to the following rules:

a. Each such mathematical symbol or expression must be preceded and followed by two blank spaces except when it occurs at the beginning or at the end of a line. Examples:

In the Middle of a Line:

The symbol x + y denotes the sum of x and y.

⠠⠮ ⠎⠽⠍⠃⠕⠇ ⠭⠬⠽ ⠙⠑⠝⠕⠞⠑⠎ ⠮ ⠎⠥⠍
⠷ ⠰⠭ ⠯ ⠰⠽⠲

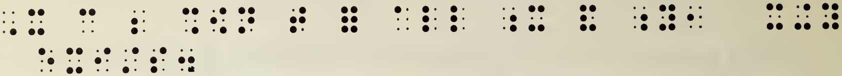
At the Beginning of a Line:

If x and y are natural numbers, x + y is also a natural number.

⠠⠊⠋ ⠰⠭ ⠯ ⠰⠽ ⠜⠑ ⠝⠁⠞⠥⠗⠁⠇ ⠝⠥⠍⠃⠻⠎⠂
⠭⠬⠽ ⠊⠎ ⠁⠇ ⠁ ⠝⠁⠞⠥⠗⠁⠇ ⠝⠥⠍⠃⠻⠲

At the End of a Line:

It can be shown that for all x and y, x + y exists.



b. When marks of punctuation are applied to mathematical expressions which must be represented by the symbols of the Nemeth Code, such marks of punctuation are represented by the signs provided for them in Standard English Braille, but they must be preceded by the [3] symbol ⠠ (dot 6). The rule for leaving two blank spaces applies even after a mark of punctuation. It is not necessary to insert dot 6 before the period following a dash representing missing numbers, although no space should be left between the dash and the period. See Section 3, page 2. (Br. ed., page 4).

c. A footnote should be added to the text whenever the symbols of the Nemeth Code are employed, and when their meaning needs to be interpreted to the reader.

2. **Mathematical Texts** — In writing literary expressions in texts which, in purpose or context, are predominantly mathematical, the symbols and usages of the Nemeth Code should be employed, except where the use of Standard English Braille would lead to less confusion, such as in the case of italicized numbers and such phrases as "for men aged 60-70," "No. 2½ can," or "5½ shoe." In texts which deal with mathematical subject matter but in which a literary format is employed (such as physics and chemistry texts, radio handbooks, and the like), a note [4] at the beginning of the text or a footnote in an appropriate place should be included to the effect that the symbols and usages of the Nemeth Code are being followed, with the following exceptions: numbers of volumes, chapters, and pages; numbers of problems or exercises; dates; and points of time, although a period of time, such as "5 minutes," should be expressed according to the Nemeth Code.

If it is necessary to interpolate a literary word or phrase into a mathematical text whose representation is in accordance with the symbols and usages of the Nemeth Code, the rule for leaving two blank spaces before and after such words or phrases should be observed.

3. **Punctuation** — In punctuating mathematical expressions, the Standard English Braille punctuation signs should be employed subject to the following rules:

a. When mathematical expressions are written in accordance with the usages of Standard English Braille, the rules regarding punctuation in that system should be observed.

b. When mathematical expressions are written in accordance with the usages of the Nemeth [5] Code, the punctuation signs are the same as those used in Standard English Braille, but such signs must be preceded by the symbol ⠠ (dot 6). See Section 1-b, page 2. (Br. ed., pages 2-3). This rule must be followed even for punctuation signs which precede mathematical expressions, such as the opening parenthesis and the opening quotation mark, both of which must be preceded by dot 6. This rule also applies when a mathematical expression is joined to a word by a hyphen, in which case the hyphen must be preceded by dot 6. For example:

⠢⠠⠤⠉⠢⠞ means 5-cent

⠂⠴⠠⠤⠏⠨⠙ means 10-pound

II

THE REPRESENTATION OF NUMBERS

See Sections 37-44, pages 18-20, (Br. ed., pages 38-44) for the representation of Roman numerals and all types of letters respectively.

In the writing of literary texts, the symbols and rules of usage provided in Standard [6] English Braille should be observed. In the writing of mathematical texts, the following symbols and usages should be employed:

4. **The Digits** — When the digits are represented within a text which is wholly or predominantly mathematical, or as part of an expression not written according to Standard English Braille, they are represented by the ten lower signs corresponding to the letters from **a** to **j**. Thus:

⠂ ⠆ ⠒ ⠲ ⠢ ⠖ ⠶ ⠦ ⠔ ⠴

represent, respectively, the digits 1 2 3 4 5 6 7 8 9 0 . For example:

⠆⠶ ⠂⠲⠢ ⠂⠔⠢⠆

represent, respectively,

27 145 1952.

⠼⠃⠛ ⠼⠁⠙⠑ ⠼⠁⠊⠑⠃⠲

When missing digits are indicated in ink print by a question mark or other device, the sign ⠿ (dots 1-2-3-4-5-6) should be employed in Braille. For example:

⠀⠲⠴⠆⠂
⠤⠿⠿⠿⠿
⠀⠒⠒⠒⠒ means $$\begin{array}{r} 4021 \\ -\ ???? \\ \hline 1578 \end{array}$$ [7]
⠀⠂⠢⠶⠦

5. **Use of the Comma in Numbers** — In numbers containing four or more digits, it is customary to insert a comma at intervals of three digits to facilitate the reading of the number. The comma is represented by the symbol ⠠ (dot 6). For example:

23,456	186,330	1,987,654.
⠆⠒⠠⠲⠢⠖	⠂⠦⠖⠠⠒⠒⠴	⠂⠠⠔⠦⠶⠠⠖⠢⠲⠠⠲

Note, in the last example above, the dot 6 preceding the period cannot be mistaken for a comma, since every comma which divides a number must be followed by exactly three digits.

6. **Division of Long Numbers** — No number should be divided at the end of a line unless it is unusually long. In no case, however, may the division be made other than after the comma. When a number is divided, the comma at the end of the line serves as sufficient indication that the number is continued on the next line. For example:

32,625,000,000,000,000,000,
000,000.

⠒⠆⠠⠖⠆⠢⠠⠴⠴⠴⠠⠴⠴⠴⠠⠴⠴⠴⠠⠴⠴⠴⠠⠴⠴⠴⠠
⠴⠴⠴⠠⠴⠴⠴⠠⠲ [8]

7. **The Decimal Point** — The decimal point is represented by the sign ⠨ (dots 4-6). For example:

3.14159	3,006.409.
⠒⠨⠂⠲⠂⠢⠔	⠒⠠⠴⠴⠖⠨⠲⠴⠔⠠⠲

8. **The Recurring Decimal** — The recurring decimal point is represented by the sign ⠸ (dots 4-5-6). It is used in addition to the usual decimal point and in the proper position. For example:

⠒⠨⠸⠒ means 3.3333 etc. ($3.\overline{3}$)
⠒⠨⠂⠸⠖ means 3.16666 etc.
⠴⠨⠸⠂⠲⠆⠦⠢⠶ means 0.142857142857 etc.

9. **The Dollar Sign ($)** — The dollar sign is represented by the sign ⠲⠼ (dots 2-5-6, 3-4-5-6). As in ink print, it precedes the quantity to which it refers. For example:

⠲⠼⠂⠴⠴ means $100
⠲⠼⠒⠨⠔⠦ means $3.98 [9]

(Note the use of the decimal sign in expressing decimal coinage).

10. **The Cent Sign (¢)** — The cent sign is represented by the symbol ⠉⠼ (dots 1-4, 3-4-5-6). This

sign should be used when indicated in the ink-print text. Contrary to ink-print usage, this sign precedes the quantity to which it refers. For example:

⠉⠼⠦ means 8¢

⠉⠼⠨⠢ means .5¢ (5/10ths of a cent)

but

⠢⠠⠤⠉⠢⠞ means 5-cent.

The sign for mills ⠍⠼ (dots 1-3-4, 3-4-5-6) is seldom used.

11. **Foreign Money**—Money of other countries should be represented by the use of the proper monetary sign preceding the numeral sign without a space. Following are symbols for some common foreign money values: [10]

Pound Sterling (£) ⠇⠼ (Dots 1-2-3, 3-4-5-6).

Shilling (/ or s) ⠎⠼ (dots 2-3-4, 3-4-5-6).

Pence (d) ⠙⠼ (dots 1-4-5, 3-4-5-6).

Guinea (gs) ⠛⠎⠼ (dots 1-2-4-5, 2-3-4, 3-4-5-6).

Franc (fr) ⠋⠗⠼ (dots 1-2-4, 1-2-3-5, 3-4-5-6).

Lira (l) ⠇⠊⠼ (dots 1-2-3, 2-4, 3-4-5-6).

Mark (Reich) (rm) ⠗⠍⠼ (dots 1-2-3-5, 1-3-4, 3-4-5-6).

Peso (P) ⠏⠼ (dots 1-2-3-4, 3-4-5-6).

Rupee (rs) ⠗⠎⠼ (dots 1-2-3-5, 2-3-4, 3-4-5-6).

Yen (Y) ⠽⠼ (dots 1-3-4-5-6, 3-4-5-6).

12. **The Per Cent Sign (%)**—The Standard English Braille per cent sign ⠒⠏ (dots 2-5, 1-2-3-4) is used when it is to be followed by a number represented according to the rules of Standard English Braille. When, [11] however, a percentage is to be represented by means of symbols provided by the Nemeth Code, the sign ⠏⠉ (dots 1-2-3-4, 1-4) is used, following the quantity to which it refers without a space. For example:

⠢⠏⠉ means 5%

⠆⠨⠔⠏⠉ means 2.9%

⠂⠆⠸⠹⠂⠌⠆⠸⠼⠏⠉ means 12½%. (See Section 28, page 13; Br. ed., page 29, for the writing of mixed numbers).

III
ABBREVIATIONS

When writing abbreviations in accordance with Standard English Braille usage, the abbreviations should be written preceding the numeral sign without a space between. In writing abbreviations according to the Nemeth Code, the abbreviation should usually (See Section 13) follow the number without a space and the numeral sign should be omitted.

13. Conformity with Ink-print Usage — When names of units are abbreviated in ink [12] print, the same abbreviations should be used in Braille. However, such abbreviations should not be contracted by the use of the Standard English Braille contractions. When, in ink print, an abbreviation follows the unit to which it applies, this is also the case in Braille. When an abbreviation precedes the quantity to which it refers in ink print, this is also the case in Braille. In either case the abbreviation and its value should be written without a space between. For example:

⠒⠋⠞ means 3 ft.

⠆⠨⠢⠲⠉⠍ means 2.54 centimeters.

It should be noted, however, that no special Braille equivalent is provided for the small, elevated single and double strokes used in ink print to represent feet and inches and minutes and seconds; rather, the abbreviations given in Section 14 should be used in their places.

14. Recommended List of Abbreviations — Any abbreviation listed in a recognized English dictionary can be used where needed, provided its use does not lead to ambiguity with other symbols of the system. In no case, [13] should the contractions for Standard English Braille be employed.

In addition to the abbreviations for monetary units and per cent given above, the following is a sample list of recommended abbreviations:

Acres (a) ⠁ (dot 1).

Barrels (bbl) ⠃⠃⠇ (dots 1-2, 1-2, 1-2-3).

Bushels (bu) ⠃⠥ (dots 1-2, 1-3-6).

Centimeters (cm) ⠉⠍ (dots 1-4, 1-3-4).

Cubic Inches (cu. in.) (These same forms can be used in writing "cubic feet," "cubic yards," etc.)

Standard English Braille Usage ⠉⠥⠲⠊⠝ (dots 1-4, 1-3-6, 2-5-6, 2-4, 1-3-4-5).

Nemeth Code ⠊⠝⠩ (in^3) (See Section 83, page 26; Br. ed., page 59, for cube sign). Examples:

⠖⠨⠲⠢⠉⠍⠩ means 6.45 cubic centimeters. (6.45 cm^3).

⠔⠦⠴⠉⠍⠌⠎⠑⠉⠣ means 980 centimeters [14] per second per second. (980 cm/sec^2).

Days (da) ⠙⠁ (dots 1-4-5, 1).

Decimeters (dm) ⠙⠍ (dots 1-4-5, 1-3-4).

Degrees (°) ⠙⠛ (dots 1-4-5, 1-2-4-5). The **Degree Sign** may be used to indicate measurements of arcs and angles as well as of temperatures.

Dozen (dz) ⠙⠵ (dots 1-4-5, 1-3-5-6).

Feet (ft or ′) ⠋⠞ (dots 1-2-4, 2-3-4-5).

Gallons (gal) ⠛⠁⠇ (dots 1-2-4-5, 1, 1-2-3).

Gills (gi) ⠛⠊ (dots 1-2-4-5, 2-4).

Grams (gm) ⠛⠍ (dots 1-2-4-5, 1-3-4).

Gross (gro) ⠛⠗⠕ (dots 1-2-4-5, 1-2-3-5, 1-3-5).

Hours (hr) ⠓⠗ (dots 1-2-5, 1-2-3-5).

Hundred Weight (cwt) ⠉⠺⠞ (dots 1-4, 2-4-5-6, 2-3-4-5).

Inches (in. or ″) ⠊⠝ (dots 2-4, 1-3-4-5). [15] The contracted form should never be used in mathematical Braille.

Kilograms (kg) ⠅⠛ (dots 1-3, 1-2-4-5).

Kilometers (km) ⠅⠍ (dots 1-3, 1-3-4).

Meters (m) ⠍ (dots 1-3-4).

Miles (mi) ⠍⠊ (dots 1-3-4, 2-4).

Millimeters (mm) ⠍⠍ (dots 1-3-4, 1-3-4).

Minutes (min or ′) ⠍⠊⠝ (dots 1-3-4, 2-4, 1-3-4-5) or ⠍⠝ (dots 1-3-4, 1-3-4-5). The former is the preferred usage, and should never be written in contracted form in mathematical Braille. This sign may be used to indicate minutes of angles and arcs as well as of time.

Months (mo) ⠍⠕ (dots 1-3-4, 1-3-5).

Ounces (oz) ⠕⠵ (dots 1-3-5, 1-3-5-6).

Pecks (pk) ⠏⠅ (dots 1-2-3-4, 1-3). [16]

Pints (pt) ⠏⠞ (dots 1-2-3-4, 2-3-4-5).

Pound Weight (# or lb) ⠇⠃ (dots 1-2-3, 1-2).

Quarters (qr) ⠟⠗ (dots 1-2-3-4-5, 1-2-3-5).

Quarts (qt) ⠟⠞ (dots 1-2-3-4-5, 2-3-4-5).

Rods (rd) ⠗⠙ (dots 1-2-3-5, 1-4-5).

Seconds (sec or ″) ⠎⠑⠉ (dots 2-3-4, 1-5, 1-4). The sign for Seconds may be used to indicate seconds of angles and arcs as well as of time. ("Sc" may not be used because of the possible confusion with the abbreviation for "since" in geometry and algebra.)

Square Feet (sq. ft.) (These same forms can be used in writing "square inches," "square yards," etc.)

Standard English Braille Usage ⠎⠟ ⠲ ⠋⠞ (dots 2-3-4, 1-2-3-4-5, 2-5-6, 1-2-4, 2-3-4-5).

Nemeth Code ⠋⠞⠣ (dots 1-2-4, 2-3-4-5, 1-2-6). (See Section 83, page 26; Br. ed., page 59, for [17] square sign). Example:

⠂⠲⠲⠋⠞⠣ means 144 square feet. (144 ft^2)

Tablespoons (tbsp) ⠞⠃⠎⠏ (dots 2-3-4-5, 1-2, 2-3-4, 1-2-3-4).

Teaspoons (tsp) ⠞⠎⠏ (dots 2-3-4-5, 2-3-4, 1-2-3-4).

Tons (t) ⠞ (dots 2-3-4-5).

Weeks (wk) ⠺⠅ (dots 2-4-5-6, 1-3).

Yards (yd) ⠽⠙ (dots 1-3-4-5-6, 1-4-5).

Years (yr) ⠽⠗ (dots 1-3-4-5-6, 1-2-3-5).

Additional **Symbols of Measurement** may be developed by placing the accepted abbreviation authorized by any standard dictionary before the numeral sign or after a number without a space. In this connection, it should be noted that the form ⠉ (dots 1-4) cannot be used for "cups" or "cans," since in Braille this character represents "cents." It is also suggested that [18] no abbreviation be used for "section," since this is not often needed and might be confused with the abbreviation for "seconds."

15. **Sequence of Related Terms** — Where Standard English Braille usage is employed, the abbreviation and unit of value for the largest quantity are written first, followed successively by the smaller quantities,

each preceded by the numeral sign and written without intervening spaces. For example:

⠙⠛⠼⠉⠑⠼⠚⠼⠙⠑ means 35°0′45″ (35 degrees, 0 minutes, 45 seconds)

⠽⠙⠼⠙⠼⠃⠼⠉ means 4 yds, 2 ft, 3 in

⠓⠗⠼⠉⠼⠃⠚⠼⠙ means 3 hrs, 20 min, 4 sec

In writing such sequences according to the Nemeth Code, the unit of value of the largest quantity and its abbreviation are written first, followed immediately without a space by the quantity of the second smaller unit, followed without spacing by the quantities of the successively smaller units each preceded by dot 3. Where [19] intervening submultiples are omitted, zeros should be inserted in their places in order to preserve the order of succession. If, however, there is any doubt as to missing submultiples, the abbreviations for each submultiple should be used in place of dot 3. For example:

⠒⠢⠙⠛⠴⠄⠲⠢ or ⠒⠢⠙⠛⠲⠢⠎⠑⠉ means 35°0′45″ (35 degrees, 0 minutes, 45 seconds)

⠲⠽⠙⠆⠄⠒ means 4 yds, 2 ft, 3 in

⠒⠓⠗⠆⠴⠄⠲ means 3 hrs, 20 min, 4 sec

An exception to these rules for both literary and mathematical Braille occurs in expressing a definite point of time when the Standard English Braille number sign and numbers should always be used and the colon (dots 2-5) should be placed between the hours and minutes without a space. For example:

⠼⠉⠒⠃⠚ means 3:20 (twenty minutes past three). [20]

IV

SIGNS OF OPERATION USED IN ARITHMETIC

For Advanced Signs of Operation, see Sections 45-48, pages 20-21,
(Br. ed., pages 44-46).

16. Plus Sign (+) ⠬ (dots 3-4-6). Examples:

⠆⠬⠒ means 2 + 3

⠭⠬⠽ means x + y

⠀⠲⠢⠖
⠬⠀⠶⠆
⠀⠒⠒⠒
⠀⠢⠆⠦

means $\begin{array}{r}456\\ +\ 72\\ \hline 528\end{array}$

17. Minus Sign (—) ⠤ (dots 3-6). Examples:

⠶⠤⠲ means 7 — 4

⠆⠁⠤⠒⠃ means 2a — 3b

⠀⠲⠢⠖
⠤⠀⠶⠆
⠀⠒⠒⠒
⠀⠒⠦⠲

means $$\begin{array}{r}456\\-\ 72\\\hline 384\end{array}$$

[21]

18. Multiplication Sign (×) ⠡ (dots 1-6). Examples:

⠒⠡⠢ means 3 × 5

⠀⠀⠲⠢⠖
⠀⠀⠡⠶⠆
⠒⠒⠒⠒⠒
⠀⠀⠔⠂⠆
⠒⠂⠔⠆
⠒⠒⠒⠒⠒
⠒⠆⠦⠒⠆

means $$\begin{array}{r}456\\\times 72\\\hline 912\\3192\ \\\hline 32832\end{array}$$

19. Fraction Line (—) ⠌ (dots 3-4). Example:

⠲⠌⠶ means $\frac{4}{7}$ (the fraction 4 over 7, whether in ink print the fraction line is represented by the horizontal line or by the diagonal stroke.)

For the manner in which fractions are represented in this system see Sections 25-30, pages 12-14, (Br. ed., pages 26-32). See also Section 48, page 21, (Br. ed., page 45) for the diagonal stroke.

20. Division Sign (÷) ⠌⠌ (dots 3-4, 3-4). Examples: [22]

⠲⠔⠌⠌⠶ means 49 ÷ 7 (49 divided by 7)

⠆⠭⠌⠌⠒⠽ means 2x ÷ 3y (2x divided by 3y)

This sign is also used to represent the main fraction line of a complex fraction in which the numerator, or the denominator, or both, are themselves fractions. For example:

⠁⠌⠃⠌⠌⠉ represents $\frac{a/b}{c}$ (the complex fraction whose numerator is the fraction a over **b**, and whose denominator is **c**).

⠁⠌⠌⠃⠌⠉ represents $\frac{a}{b/c}$ (the complex fraction whose numerator is **a**, and whose denominator is the fraction **b/c**).

⠁⠌⠃⠌⠌⠉⠌⠙ represents $\frac{a/b}{c/d}$ (the complex fraction whose numerator is the fraction **a/b**, and whose denominator is the fraction **c/d**).

It is seldom necessary to distinguish between the division sign and the main fraction line of a complex fraction. If, however, it should become necessary or desirable to do so, the sign ⠌⠌ (dots 3-4, 3-4) retains its meaning as the main fraction line of a complex fraction, while the division sign, which is of less frequent [23] occurrence, is represented by the sign ⠸⠌⠌ (dots 4-5-6, 3-4, 3-4).

21. The Sign for "Divided Into" ($)\overline{\quad}($) — This is a sign used particularly in arithmetic. The sign ⠕ (dots 1-3-5) is used to separate the divisor from the dividend, while the sign ⠪ (dots 2-4-6) is placed after the dividend and may be followed by the quotient, and a space should be left between the quotient and any remainder. For example:

⠂⠶⠕⠆⠦⠔⠪ means $17)\overline{289}($

⠦⠕⠖⠴⠪⠶⠨⠢ means $8)\overline{60}(7.5$

⠢⠕⠂⠖⠪⠒ ⠗⠂ means $5)\overline{16}(3$ r1

or

⠪ ⠒ ⠗⠂
⠢⠕⠂⠖ means $\begin{array}{r} 3\ \text{r}1 \\ 5)\overline{16\quad} \end{array}$

Note: As the examples of this section indicate, no space is left before or after the signs of operation except when representing [24] functions having special names, for which see Chapter XX.

V

SYMBOLS OF COMPARISON USED IN ARITHMETIC

See also Sections 52-75, pages 23-24, (Br. ed., pages 50-54) for Advanced Symbols of Comparison.

In writing symbols of comparison, it should be noted that blank spaces are always left before and after them.

22. **Equality Sign** (=) ⠨⠅ (dots 4-6, 1-3). Example:

⠆⠬⠆ ⠨⠅ ⠲ means $2 + 2 = 4$

23. "Is To" (:) ⠐⠂ (dots 5, 2). Example:

⠂ ⠐⠂ ⠆ ⠨⠅ ⠒ ⠐⠂ ⠖ means $1 : 2 = 3 : 6$

24. "As" (: :) — The double colon in ink print represents "as" when used in a proportion. In Braille, it is sufficient to express this symbol by the equality sign as in the [25] above expression. If, however, it becomes necessary or desirable to express this symbol explicitly, the symbol ⠰⠆ (dots 5-6, 2-3) is used. Thus, the above example may be written as follows:

⠂ ⠐⠂ ⠆ ⠰⠆ ⠒ ⠐⠂ ⠖ means $1 : 2 :: 3 : 6$

VI

REPRESENTATION OF FRACTIONS

We have already seen that the fraction line is represented by the sign ⠌ (dots 3-4).

25. **Simple Fractions** — When a fraction contains a single term in its numerator and a single term in its denominator, it is represented simply by writing the numerator and the denominator separated by the fraction line. For example:

⠆⠌⠒ means $2/3$ (two-thirds)

⠭⠌⠢ means $x/5$ (x over 5)

⠁⠭⠌⠃ means $\frac{ax}{b}$ (ax over b) [26]

⠒⠌⠭⠽ means $\frac{3}{xy}$ (3 over xy)

⠆⠁⠃⠉⠌⠭⠽⠵ means $\frac{2abc}{xyz}$ (2abc over xyz)

If the numerator or the denominator of a fraction requires more than four Braille cells, and one of the characters is not a number, the extent of the fraction must be indicated by the fraction indicator signs as shown in Section 26. For example:

⠹⠪⠭⠤⠽⠕⠪⠭⠬⠽⠕⠌⠭⠣⠼ means $\frac{(x - y)\ (x + y)}{x^2}$

⠹⠒⠌⠪⠁⠬⠃⠕⠪⠆⠁⠤⠒⠃⠕⠼ means $\frac{3}{(a + b)\ (2a - 3b)}$

26. Showing the Extent of a Fraction — If the numerator, or the denominator, or both, are expressions consisting of the sum of terms, it is necessary to show where the fraction begins and ends. To accomplish this, the symbol ⠹ [27] (dots 1-4-5-6) precedes the fraction, and the symbol ⠼ (dots 3-4-5-6) follows the fraction. Thus, every mathematical symbol which follows the first of these signs and precedes the fraction line belongs to the numerator of the fraction, while every mathematical symbol which follows the fraction line and precedes the second of the above symbols belongs to the denominator of the fraction. For example:

⠹⠁⠬⠭⠌⠁⠤⠭⠼ means $\frac{a + x}{a - x}$

⠹⠁⠬⠃⠌⠉⠼ means $\frac{a + b}{c}$

⠁⠬⠃⠌⠉ means $a + \frac{b}{c}$

⠁⠌⠃⠬⠉ means $\frac{a}{b} + c$

The fraction line which may occur within an exponent does not terminate the numerator of the main fraction. For example: [28]

⠹⠭⠘⠆⠌⠒⠐⠬⠽⠘⠆⠌⠒⠐⠌⠵⠘⠆⠌⠒⠐⠼ means $\frac{x^{2/3} + y^{2/3}}{z^{2/3}}$

27. Representing the Product of a Fraction and a Whole Number — The signs which indicate the beginning and end of a fraction must sometimes be used even when writing a simple fraction, as is the case when indicating the product of such a fraction and a whole number. However, it is not necessary to use both of these signs when one of them is used, when the meaning is otherwise clear. For example:

⠭⠹⠽⠌⠵ means $x\frac{y}{z}$

⠏⠌⠟⠼⠗ means $\frac{p}{q}r$

Fractional coefficients are represented in this way. For example:

⠒⠌⠢⠼⠭ means $\frac{3}{5}x$ [29]

28. Mixed Numbers — When writing a mixed number, the sign ⠸⠹ (dots 4-5-6, 1-4-5-6) separates the integral part from the fractional part. When the fractional part is followed by another character, except a mark of punctuation, it is necessary to insert the sign ⠸⠼ (dots 4-5-6, 3-4-5-6) before the next character. For example:

⠂⠸⠹⠂⠌⠆ means 1½

⠆⠸⠹⠆⠌⠒⠸⠼⠏⠉⠬⠒⠸⠹⠂⠌⠲⠸⠼⠏⠉ means 2-2/3% + 3-1/4%

29. Complex Fractions — Complex fractions are fractions whose numerators, or denominators, or both, are themselves fractional. In representing complex fractions, the main fraction line must be represented by the sign ⠌ ⠌ (dots 3-4, 3-4). For example:

⠁⠌⠃⠌⠌⠉⠌⠙ means $\frac{a/b}{c/d}$ [30]

When it is necessary or desirable to show that a fraction is complex, the sign ⠠ ⠹ (dots 6, 1-4-5-6) precedes such a fraction, while the sign ⠠ ⠼ (dots 6, 3-4-5-6) follows it. For example:

⠠⠹⠹⠁⠬⠃⠌⠉⠬⠙⠼⠌⠌⠹⠑⠬⠋⠌⠛⠬⠓⠼⠠⠼ means $\frac{\frac{a+b}{c+d}}{\frac{e+f}{g+h}}$

(the complex fraction whose numerator is **a** + **b** over **c** + **d**, and whose denominator is **e** + **f** over **g** + **h**).

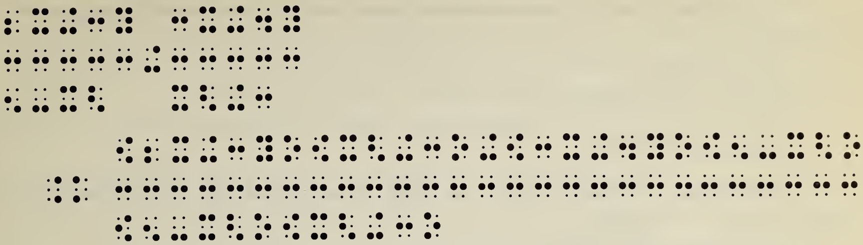
In ink print, the main fraction line of a complex fraction is sometimes not distinguishable from the partial fraction line. In such cases, the identity of the main fraction line is clear because of the coefficient, symbol of comparison, or other mathematical symbol which stands next to such a fraction. When the main fraction line is distinguished in ink print, this is accomplished by printing it in heavier type. In Braille, the main fraction line must be made explicit even when this is not the case in ink print. For example: **[31]**

⠭⠹⠁⠌⠌⠃⠌⠉⠼ means $x\frac{a}{\frac{b}{c}}$

⠭⠹⠁⠌⠃⠌⠌⠉⠼ means $x\frac{\frac{a}{b}}{c}$

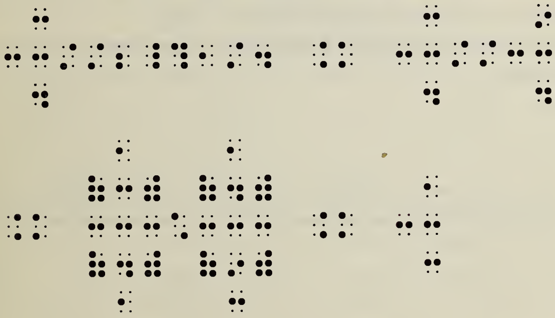
⠁⠌⠌⠃⠌⠉ ⠨⠌⠅ ⠁⠌⠃⠌⠌⠉ means $\frac{a}{\frac{b}{c}} \neq \frac{\frac{a}{b}}{c}$

30. The Spatial Representation of Fractions, Cancellation — It is sometimes desirable to represent fractions in the same arrangement as is used for this purpose in ink print. In that case, a sequence of the sign ⠒ (dots 2-5) is used, and extends the length of the fraction. For example:



means $\frac{2x + 3y}{5 - x^2} + \frac{3x + 4y}{x^2 + 3} = \frac{(2x + 3y)(x^2 + 3) + (3x + 4y)(5 - x^2)}{(5 - x^2)(x^2 + 3)}$ [32]

When illustrating cancellation, the fractions must always be represented spatially and the quantities to be cancelled out should be enclosed in brackets. For example:

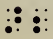


means $\frac{3}{4} \div 2\frac{1}{4} = \frac{3}{4} \div \frac{9}{4} = \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{3}{\cancel{9}}} = \frac{1}{3}$

VII

THE REPRESENTATION OF RADICALS

31. The Radical Sign—The radical sign is represented by the sign ⠜ (dots 3-4-5). As is the case in ink print, when this sign is otherwise unmodified, it represents the square root sign. For example: [33]

 means $\sqrt{2}$

The radical sign may be left unterminated only when it applies to a single symbol or to a positive integer. For example:

⠜⠧ means $\sqrt{v}$

⠜⠂⠒ means $\sqrt{13}$

⠜⠢⠭ means $\sqrt{5}x$

32. **Terminating the Radical Sign** — The effect of the radical sign is terminated by the sign ⠻ (dots 1-2-4-5-6). For example:

⠜⠢⠭⠻ means $\sqrt{5x}$

⠜⠁⠭⠣⠬⠃⠭⠬⠉⠻ means $\sqrt{ax^2 + bx + c}$

⠭ ⠨⠅ ⠹⠤⠃⠬⠤⠜⠃⠣⠤⠲⠁⠉⠻⠌⠆⠁⠼ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

is the formula for the solution of the general quadratic equation. [34]

When the radical sign applies to a fraction, the extent of the fraction must always be indicated even when it is a simple fraction. For example:

⠜⠹⠭⠌⠽⠼⠻ means $\sqrt{x/y}$

⠜⠭⠌⠽ means $\frac{\sqrt{x}}{y}$

33. **The Index of a Radical** — If a radical has an index other than 2, this must be made explicit in Braille as is the case in ink print. The index of the radical precedes the radical sign and is preceded by the sign ⠸ (dots 4-5-6). For example:

⠸⠒⠜⠆ means $\sqrt[3]{2}$

⠒⠜⠆ means $3\sqrt{2}$

⠸⠁⠅⠬⠃⠍⠜⠭ means $\sqrt[ak+bm]{x}$

⠸⠢⠜⠒⠆⠤⠭⠣⠻ means $\sqrt[5]{32 - x^2}$ [35]

⠸⠁⠬⠃⠜⠭⠤⠽⠻ means $\sqrt[a+b]{x - y}$

⠸⠏⠜⠭ means $\sqrt[p]{x}$

⠨⠏⠜⠭ means $\pi\sqrt{x}$

34. **Radicals Under Radicals** — When it is necessary to represent radicals under radicals, the largest

radical is represented in the manner described in the preceding sections. The next smaller radical is represented by the sign ⠸⠜ (dots 4-5-6, 3-4-5). The third radical is represented by the sign ⠨⠜ (dots 4-6, 3-4-5). Radicals smaller than this are rare and may be represented by the sign ⠸⠸⠜ (dots 4-5-6, 4-5-6, 3-4-5), or by the sign ⠨⠨⠜ (dots 4-6, 4-6, 3-4-5). The radical termination signs must be preceded by the same indicators as are used to introduce the radicals. For example:

⠜⠭⠬⠜⠽ means $\sqrt{x} + \sqrt{y}$ [36]

⠜⠭⠬⠸⠜⠽⠸⠻⠻ means $\sqrt{x + \sqrt{y}}$

⠸⠒⠜⠆⠬⠜⠢ means $\sqrt[3]{2} + \sqrt{5}$

⠸⠒⠜⠆⠬⠸⠜⠢⠸⠻⠻ means $\sqrt[3]{2 + \sqrt{5}}$

⠸⠒⠜⠆⠭⠣⠬⠦⠽⠣⠬⠸⠜⠒⠭⠽⠸⠻⠤⠵⠣⠻
means $\sqrt[3]{2x^2 + 8y^2 + \sqrt{3xy} - z^2}$

It is often useful to enclose inner radicals by appropriate symbols of grouping. In that case, the radical termination sign is unnecessary, since the symbols of grouping specify the effect of the radical. Thus, the preceding example may be written as:

⠸⠒⠜⠆⠭⠣⠬⠦⠽⠣⠬⠸⠜⠪⠒⠭⠽⠕⠤⠵⠣⠻⠠⠲
$\sqrt[3]{2x^2 + 8y^2 + \sqrt{3xy} - z^2}$ [37]

35. **Large Radical Signs** — When the radical sign in ink print is larger than usual, as is the case, for example, when it concerns a fraction, it is generally unnecessary in Braille to indicate that the radical sign in question is larger than usual. If, however, it becomes necessary or desirable to show that a radical sign is large, the radical sign itself, as well as the termination sign, is prefixed by the sign ⠠ (dot 6). For example:

⠠⠜⠹⠭⠬⠽⠌⠭⠤⠽⠼⠠⠻ means $\sqrt{\frac{x + y}{x - y}}$ and specifies that in ink print the radical sign is large.

VIII

PI

36. The ratio of the circumference of a circle to its diameter is represented by the Greek letter Pi (π), and has a numerical [38] value of approximately 3.1416. In this system, it is represented by ⠨⠏ (dots 4-6, 1-2-3-4), and is written without a space before or after it.

IX

REPRESENTATION OF LETTERS

37. **Small Roman Letters or Numerals** — Letters of the Roman alphabet are represented in this system by the same signs which represent them in Standard English Braille. For example:

⠆⠭ means 2x

⠶⠁⠃ means 7ab

38. **Capitalized Roman Letters or Numerals** — A capitalized Roman letter is represented by prefixing the sign ⠠ (dot 6) to the corresponding small Roman letter. When it is necessary to represent a sequence of capitalized Roman letters, each such letter must be preceded by the prefix for capitalization. For example: [39]

⠠⠭⠠⠽ means XY

⠒⠠⠍⠠⠝ means 3MN

⠠⠗⠎⠠⠞ means RsT

39. **Small Greek Letters** — Small Greek letters are represented in this system by the signs of the International Greek Alphabet prefixed by the sign ⠨ (dots 4-6). The alphabet of small Greek letters as represented in this system, plus also the ink-print capital and small letter Greek alphabet, follow:

Alpha	Α	α	⠨⠁
Beta	Β	β	⠨⠃
Gamma	Γ	γ	⠨⠛
Delta	Δ	δ	⠨⠙
Epsilon	Ε	ϵ	⠨⠑
Zeta	Ζ	ζ	⠨⠵
Eta	Η	η	⠨⠱
Theta	Θ	θ	⠨⠹
Iota	Ι	ι	⠨⠊
Kappa	Κ	κ	⠨⠅
Lambda	Λ	λ	⠨⠇
Mu	Μ	μ	⠨⠍
Nu	Ν	ν	⠨⠝
Xi	Ξ	ξ	⠨⠭
Omicron	Ο	ο	⠨⠕
Pi	Π	π	⠨⠏
Rho	Ρ	ρ	⠨⠗
Sigma	Σ	σ or ς	⠨⠎
Tau	Τ	τ	⠨⠞
Upsilon	ϒ	υ	⠨⠥

Phi	Φ	ϕ	⠨⠋	Psi	Ψ	ψ	⠨⠽
Chi	X	χ	⠨⠯	Omega	Ω	ω	⠨⠺ [40]

It should be noted that in ink print there are two representations of the small Greek sigma. In Braille, it is not necessary to distinguish between them, except when it becomes necessary or desirable to do so. In that case, the more common sigma (σ) retains its representation as listed above, while the sigma which has a form resembling the small Roman s (ς) is represented by the sign ⠨⠮ (dots 4-6, 2-3-4-6).

40. **Capitalized Greek Letters** — Capitalized Greek letters are represented in this system by their signs from the International Greek Alphabet (see Section 39 above) and bearing the indicator ⠨⠠ (dots 4-6, 6). For example:

⠨⠠⠙ represents the capitalized delta (Δ)

⠨⠠⠎ represents the capitalized sigma (Σ)

41. **Letters from Alphabets Other Than the Roman or Greek** — It is often necessary to represent letters from alphabets other than the Roman or Greek. When this is the case, such letters are represented by the signs of the International Alphabet to which they belong, preceded by the sign ⠸ (dots 4-5-6). Example: [41]

⠸⠁ represents the German a

Since the prefix ⠸ (dots 4-5-6) is used in several connections, a transcriber's footnote should be contained in the text to specify the exact ink-print symbol which is being represented.

When such letters are capitalized, they are prefixed by the sign ⠸⠠ (dots 4-5-6, 6).

42. **Roman Letters Having a Form Other Than the Usual Form** — It is quite often the case that Roman letters appear in forms other than their usual printed forms. Thus, it sometimes becomes necessary to represent script letters, letters written in boldface type, or letters written in Gothic style. In such cases, these letters are represented as small Roman letters prefixed by the sign ⠸ (dots 4-5-6). For example:

⠸⠑ represents a script *e*

⠸⠙ represents a **d** written in boldface type [42]

⠸⠍ represents an 𝔪 written in Gothic style

As above, a transcriber's footnote should be contained in the text to specify the exact ink-print symbol which is being represented.

43. **Representing Vectors** — Vectors are represented by Roman letters prefixed by the sign ⠸ (dots

4-5-6). In ink print, vectors are represented by letters in boldface type and this representation is in accordance with the provisions of the preceding paragraph. However, since vectors are of relatively common occurrence, it is not necessary to include a footnote in this case. The Braille indicator ⠸ (dots 4-5-6) with no other explanation is understood to represent a vector.

44. **Resolving Possible Ambiguity Resulting from the Use of the Braille Indicator ⠸ (dots 4-5-6) —** The practice of using the Braille indicator ⠸ (dots 4-5-6) to distinguish between the various types of letters described above should not, in general, lead to confusion since, in any particular mathematical [43] discussion, only single letters of this kind or single alphabets are used. Except in the representation of vectors, a footnote should be included to specify the exact ink-print symbol which is being represented.

If a mathematical discussion requires the representation of two distinct ink-print symbols which, in Braille, would appear alike due to the use of the symbol ⠸ (dots 4-5-6), one of these symbols must be represented in some other manner. To do this, we may use the symbol ⠠⠠ (dots 6, 6), or the sign ⠨⠨ (dots 4-6, 4-6), or the sign ⠸⠸ (dots 4-5-6, 4-5-6). The choice of sign to be used to distinguish between two such symbols rests with the transcriber, but the meaning of such representations must be made explicit in a footnote. For example:

> If a script ℯ and a Gothic style 𝔢 appear in the same mathematical discussion, the script ℯ may be represented by the symbol ⠸⠑ while the Gothic style 𝔢 may be represented by the [44] symbol ⠨⠨⠑.

The meaning of both representations must be made explicit in a footnote.

X

SIGNS OF OPERATION USED IN ALGEBRA AND ADVANCED MATHEMATICS

See also Sections 16-21, pages 9-11, (Br. ed., pages 20-23) for signs of operation used in arithmetic.

As is the case with signs of operation used in arithmetic, no space is left before or after these signs, except when representing functions having **special names**, for which see Chapter XX.

45. **Plus or Minus** (±) ⠬⠤ (dots 3-4-6, 3-6). Example:

⠒⠁⠬⠤⠆⠃ means 3a ± 2b

46. **Minus or Plus** (∓) ⠤⠬ (dots 3-6, 3-4-6). Example: [45]

⠲⠃⠤⠬⠂ means 4b ∓ 1

47. **Multiplication Sign When Represented in Ink Print by a Dot** (.) — The multiplication sign in ink print is represented either by the "cross", see Section 18, page 10, (Br. ed., page 21), or the "dot." A single Braille sign ⠡ (dots 1-6) is used to represent the multiplication sign without distinguishing between the alternative ink-print representations, where both are not present in the same material. However, it sometimes becomes necessary or desirable to distinguish between the "dot" and the "cross." In that case, the sign ⠡ (dots 1-6) is used to represent the "cross," while the sign ⠡⠡ (dots 1-6, 1-6) is used to represent the "dot." The larger sign has been chosen to represent the "dot" because it is of less frequent occurrence than the "cross."

48. **Fraction Line When Represented in Ink Print by a Diagonal Stroke** (/) — The fraction line in ink print may be represented [46] either by a horizontal line which separates the numerator of a fraction from its denominator, see Section 19, page 10, (Br. ed., page 21), or the diagonal stroke which serves the same purpose. A single Braille sign ⠌ (dots 3-4) may be used in either case. If, however, it should become necessary or desirable to distinguish between the horizontal line and the diagonal stroke, then the sign ⠌ (dots 3-4) is used to represent the horizontal line, while the sign ⠸⠌ (dots 4-5-6, 3-4) is used to represent the diagonal stroke.

XI

SYMBOLS OF GROUPING USED IN WRITING ALGEBRA

See also Sections 99-101, pages 32-33, (Br. ed., pages 71-73) for the
representation of additional symbols in advanced mathematics.

Symbols of grouping are necessary in the writing of algebra and other more advanced branches of mathematics. As the examples in this Chapter indicate, no space is left before or after [47] any of the symbols of grouping.

49. **Parentheses** ( ) — The opening parenthesis is represented by the sign ⠪ (dots 2-4-6). The closing parenthesis is represented by the sign ⠕ (dots 1-3-5). For example:

⠁⠪⠃⠤⠉⠕ means a(b — c)

⠪⠭⠤⠽⠕⠪⠭⠬⠽⠕ means (x — y) (x + y)

When, in ink print, parentheses extend over more than one line of writing, as is the case, for example, when enclosing fractions, it is not generally necessary to indicate in Braille that the parentheses in question are larger than usual. If, however, it should become desirable or necessary to distinguish between small and large parentheses, then the small parentheses are represented as indicated above, while the large opening parenthesis and the large closing parenthesis are represented respectively by the signs ⠠⠪ (dots 6, 2-4-6) and ⠠⠕ (dots 6, 1-3-5). [48]

50. **Brackets** [ ] — The opening bracket is represented by the sign ⠷ (dots 1-2-3-5-6). The closing bracket is represented by the sign ⠾ (dots 2-3-4-5-6). For example:

⠆⠷⠁⠤⠪⠃⠤⠉⠕⠾ means 2[a — (b — c)]

When in ink print, brackets extend over more than one line of writing, as is the case, for example, when enclosing fractions, it is not generally necessary to indicate in Braille that the brackets in question are larger than usual. If, however, it becomes necessary or desirable to make such a distinction, the small brackets are represented as indicated above, while the large opening bracket and the large closing bracket are represented respectively by the signs ⠠⠷ (dots 6, 1-2-3-5-6) and ⠠⠾ (dots 6, 2-3-4-5-6).

It is not generally necessary to distinguish between square [ ] and curly { } brackets, and the bracket signs as indicated above are used to represent either one. When, however, it becomes necessary or desirable [49] to make such a distinction, the curly opening bracket and the curly closing bracket are represented respectively by the signs ⠸⠷ (dots 4-5-6, 1-2 3-5-6) and ⠸⠾ (dots 4-5-6, 2-3-4-5-6). A distinction of this kind is often made in ink print when the need arises for a third set of symbols of grouping.

If it becomes necessary or desirable to distinguish between large square brackets and large curly brackets, the large square brackets are represented as indicated above, while the large curly opening bracket and the large curly closing bracket are represented respectively by the signs ⠸⠠⠷ (dots 4-5-6, 6, 1-2-3-5-6) and ⠸⠠⠾ (dots 4-5-6, 6, 2-3-4-5-6).

51. **The Vinculum** — The vinculum in ink print is a straight line drawn above an expression to indicate that it is to be regarded as a single number and, in this respect, performs the function of a pair of symbols of grouping. It is generally unnecessary to represent the vinculum in Braille, since it may be replaced [50] advantageously by appropriate symbols of grouping. If, however, it should become necessary or desirable, this is achieved by using the sign ⠸⠳ (dots 4-5-6, 1-2-5-6) before the first symbol of the quantity under the vinculum and after the last symbol of this quantity. For example:

It is better to write ⠁⠤⠪⠃⠤⠉⠕ than ⠁⠤⠸⠳⠃⠤⠉⠸⠳

meaning $a - \overline{b - c}$

XII

SYMBOLS OF COMPARISON USED IN ALGEBRA AND ADVANCED MATHEMATICS

See also Sections 22-24, page 12, (Br. ed., pages 24) for symbols of grouping employed in arithmetic.

As the examples in this section indicate, the symbols of comparison must be preceded and followed by a blank space except when they appear in the subscript level of writing. Then, they are neither preceded nor followed by a blank space. [51]

52. **"Is Not Equal To"** ($\neq$) ⠨⠌⠅ (dots 4-6, 3-4, 1-3). Example:

⠁ ⠨⠌⠅ ⠃ means $a \neq b$

⠗ ⠨⠌⠅ ⠴ means $r \neq o$

53. **"Is Identical With"** ($\equiv$) ⠸⠇ (dots 4-5-6, 1-2-3). Example:

⠪⠁⠤⠃⠕⠪⠁⠬⠃⠕ ⠸⠇ ⠁⠣⠤⠃⠣ means $(a - b)(a + b) \equiv a^2 - b^2$

In ink print, this sign is also used to represent **"Is Congruent To"** and is used for the same purpose in Braille.

54. **"Is Not Identical With"** ($\not\equiv$) ⠸⠌⠇ (dots 4-5-6, 3-4, 1-2-3). As in ink print, this sign may also be used to represent **"Is Not Congruent To."**

55. **"Is Equivalent To"** ($\asymp$) ⠾⠷ (dots 2-3-4-5-6, 1-2-3-5-6).

56. **"Is Not Equivalent To"** ($\not\asymp$) ⠾⠌⠷ [52] (dots 2-3-4-5-6, 3-4, 1-2-3-5-6).

57. **"Is Approximately Equal To"** ($\approx$) ⠨⠡⠅ (dots 4-6, 1-6, 1-3).

58. **"Is Greater Than"** ($>$) ⠨⠂ (dots 4-6, 2).

59. **"Is Not Greater Than"** ($\ngtr$) ⠌⠨⠂ (dots 3-4, 4-6, 2).

60. **"Is Greater Than or Equal To"** ($\geqslant$) ⠨⠂⠨⠅ (dots 4-6, 2, 4-6, 1-3).

61. **"Is Equal To or Greater Than"** ($\eqslantgtr$) ⠨⠅⠨⠂ (dots 4-6, 1-3, 4-6, 2).

62. **"Is Not Greater Than or Equal To"** ($\ngeqslant$) ⠌⠨⠂⠨⠅ (dots 3-4, 4-6, 2, 4-6, 1-3).

63. **"Is Not Equal To or Greater Than"** ($\not\eqslantgtr$) ⠨⠌⠅⠨⠂ (dots 4-6, 3-4, 1-3, 4-6, 2).

64. "Is Less Than" ($<$) ⠐⠅ (dots 5, 1-3).

65. "Is Not Less Than" ($\nless$) ⠌⠐⠅ (dots 3-4, 5, 1-3).

66. "Is Less Than or Equal To" ($\leqslant$) ⠐⠅⠨⠅ (dots 5, 1-3, 4-6, [53] 1-3).

67. "Is Equal To or Less Than" ($\eqslantless$) ⠨⠅⠐⠅ (dots 4-6, 1-3, 5, 1-3).

68. "Is Not Less Than or Equal To" ($\nleqslant$) ⠌⠐⠅⠨⠅ (dots 3-4, 5, 1-3, 4-6, 1-3).

69. "Is Not Equal To or Less Than" ($\not\eqslantless$) ⠨⠌⠅⠐⠅ (dots 4-6, 3-4, 1-3, 5, 1-3).

70. "Is Greater Than or Less Than" ($><$) ⠨⠂⠐⠅ (dots 4-6, 2, 5, 1-3).

71. "Is Less Than or Greater Than" ($<>$) ⠐⠅⠨⠂ (dots 5, 1-3, 4-6, 2).

72. "Is Greater Than, Equal To, or Less Than" ($> = <$) ⠨⠂⠨⠅⠐⠅ (dots 4-6, 2, 4-6, 1-3, 5, 1-3).

73. "Is Less Than, Equal To, or Greater Than" ($< = >$) ⠐⠅⠨⠅⠨⠂ (dots 5, 1-3, 4-6, 1-3, 4-6, 2).

74. "Varies As" ($\propto$) ⠸⠿ (dots 4-5-6, 1-2-3-4-5-6). Example: [54]

⠭ ⠸⠿ ⠽ means $x \propto y$

⠭ ⠸⠿ ⠂⠌⠽ means $x \propto \frac{1}{y}$

75. "Approaches" (→) — When it is required that this symbol appear as a subscript, as, for example, when writing expressions involving limits, it is represented by the sign ⠠⠄ (dots 6, 3). When the symbol for "approaches" is not written as a subscript, it is represented by the sign ⠒⠒⠕ (dots 2-5, 2-5, 1-3-5). For example:

⠭ ⠒⠒⠕ ⠴ means $x \to 0$

⠇⠊⠍⠰⠭⠠⠄⠒⠐⠭⠣ ⠨⠅ ⠔ means $\lim_{x \to 3} x^2 = 9$ [55]

XIII

THE REPRESENTATION OF OTHER MATHEMATICAL SIGNS USED IN ALGEBRA

See also Sections 102-109, pages 33-35, (Br. ed., pages 74-78) for other signs employed in advanced branches of mathematics.

76. **Primes** (′) — The ink-print stroke which represents the prime is represented in Braille by the

sign ⠄ (dot 3). The prime sign may not be separated from the quantity to which it applies by any other symbol. If, in ink print, the prime is doubled or tripled, this is represented in Braille by doubling or tripling the prime sign respectively. For example:

⠭⠄ means x′
⠭⠄⠣ means $(x')^2$
⠋⠄⠄ means f″
⠨⠋⠄⠄⠄ means ϕ''' [56]

77. **The Mathematical Comma** — The mathematical comma is represented by the sign ⠠ (dot 6). The mathematical comma used in the separation of large numbers into periods of three digits has already been described and illustrated in Section 5, page 4, (Br. ed., page 7).

Another use for the mathematical comma is in the writing of sequences. It then follows each term without a space, but is itself followed by a blank space. For example:

⠪⠭⠠ ⠽⠕ means the point where coordinates are x and y in that order, e.g., (x, y)
⠋⠪⠭⠠ ⠽⠠ ⠵⠕ means the function of the three variables x, y, and z, e.g. $f(x, y, z)$

78. **Mathematical Ellipses** — Mathematical ellipses in ink print are represented by a group of three dots to show that intervening terms of a sequence have been omitted and that they may be formed in accordance with some rule expressed or exhibited in the sequence. In Braille, mathematical ellipses are represented by the sign ⠠⠠⠠ (dots 6, 6, 6) [57] always preceded and followed by a blank space. For example:

The harmonic sequence is formed from the reciprocals of the positive integers and is 1, 1/2, 1/3, 1/4, 1/5, 1/6 . . . 1/n.

⠂⠠ ⠂⠌⠆⠠ ⠂⠌⠒⠠ ⠂⠌⠲⠠ ⠂⠌⠢⠠
⠂⠌⠖⠠ ⠠⠠⠠ ⠂⠌⠝⠠⠲

79. **"Infinity" (∞)** — The sign for "infinity" is represented by the sign ⠿ (dots 1-2-3-4-5-6). For example:

⠂⠌⠭ ⠨⠅ ⠿ ⠱⠢ ⠭ ⠨⠅ ⠴ means $1/x = \infty$ when $x = 0$.

XIV

THE REPRESENTATION OF EXPONENTS

80. **Level-Indicating Symbols** — The writing of mathematics in ink print does not, in general, proceed in a continuous line of writing. Symbols which are written at a level above that of the base line of writing are called superscripts. They serve as [58] exponents and in other capacities.

In Braille, it is not practical to write on various levels. Consequently, this system provides signs called "level-indicating symbols" which specify the ink-print level at which mathematical symbols appear.

81. **The First Exponential Level** — To show that a mathematical symbol is a superscript at the first exponential level, the sign ⠘ (dots 4-5) is used. Every symbol which follows this sign then belongs to the first exponential level and only another level-indicating sign or the blank space which precedes a symbol of comparison may terminate the force of this sign. For example:

⠭⠘⠍ means x^m

⠭⠘⠏⠬⠟ means x^{p+q}

⠭⠘⠂⠌⠆ means $x^{1/2}$

⠭⠘⠤⠗ means x^{-r}

⠭⠘⠲ ⠨⠅ ⠭⠣⠡⠭⠣ means $x^4 = x^2 \times x^2$

82. **The Base Line of Writing** — The symbol which indicates the base line of writing [59] is ⠐ (dot 5). It is not used when the first symbol of a mathematical expression appears on the base line of writing but must be used to indicate a return to this level from another level. For example:

⠭⠘⠏⠐⠬⠭⠘⠟ means $x^p + x^q$

⠽⠘⠍⠐⠡⠽⠘⠝ ⠨⠅ ⠽⠘⠍⠬⠝ is read: $y^m \times y^n = y^{m+n}$

⠭⠘⠆⠌⠒⠐⠬⠽⠘⠆⠌⠒ ⠨⠅ ⠁⠘⠆⠌⠒ is read: $x^{2/3} + y^{2/3} = a^{2/3}$

83. **The Square and the Cube** — The powers ^{2} and ^{3} are so commonly used that special symbols have been provided for them in this system. The sign which indicates the second power is ⠣ (dots 1-2-6), and the sign which indicates the third power is ⠩ (dots 1-4-6). These symbols are not preceded by the level-indicating symbol for the first exponential [60] level nor are they followed by the level indicating symbol for the base line of writing. For example:

⠭⠣ means x^2

⠽⠩ means y^3

⠭⠩⠤⠽⠩ means $x^3 - y^3$

84. **The Second Exponential Level** — The symbol which indicates the second exponential level is ⠈ (dot 4). Every symbol which follows this sign then belongs to the second exponential level and only another level-indicating symbol or a space preceding a symbol of comparison may terminate the force of this sign. For example:

⠑⠘⠑⠈⠭ means $e^{e^{x}}$

⠑⠘⠑⠈⠭⠐⠬⠆ means $e^{e^{x}} + 2$

⠑⠘⠑⠈⠭⠘⠬⠆ means $e^{e^{x}+2}$

⠑⠘⠑⠈⠭⠬⠆ means $e^{e^{x+2}}$ **[61]**

85. **The Third Exponential Level** — When necessary, the third exponential level is represented by the sign ⠈⠈ (dots 4, 4). Like the other level-indicating symbols, its effect is carried to all the signs which follow it and is terminated only by another level-indicating symbol or by a blank space which precedes a symbol of comparison. For example:

⠑⠘⠑⠈⠑⠈⠈⠭ is read $e^{e^{e^{x}}}$

XV

THE REPRESENTATION OF SUBSCRIPTS

86. **Subscripts** — Mathematical symbols which appear below the level of the base line of writing are called subscripts. As in the case of exponents, level-indicating symbols are provided to specify the various subscript levels.

87. **The First Subscript Level** — To indicate that mathematical symbols appear **[62]** one level below the base line of writing, the symbol ⠰ (dots 5-6) is used. Its effect is carried to all the symbols which follow it and is terminated only by another level-indicating symbol or by a blank space which precedes a symbol of comparison. For example:

⠭⠰⠁ means x_a

⠭⠰⠊⠤⠂ means x_{i-1}

⠭⠰⠁⠐⠬⠭⠰⠃ means $x_a + x_b$

88. **Numerical Subscripts** — Numerical subscripts may be indicated simply by writing the subscript after the quantity to which it refers without the use of the subscript level-indicating symbol and without the use of the level-indicating symbol for the base line of writing. For example:

⠭⠴ means x_0

⠽⠂ means y_1

⠭⠂⠬⠭⠆⠬⠭⠒ means $x_1 + x_2 + x_3$ [63]

If any confusion results from this practice, numerical subscripts may be represented by the use of the appropriate level-indicating symbols in accordance with the rules specified above.

89. **The Second Subscript Level** — When it is necessary to represent subscripts two levels below the base line of writing, this is achieved by the use of the symbol ⠰⠰ (dots 5-6, 5-6). For example:

⠭⠰⠅⠰⠰⠍ means x_{k_m}

When subscripts at the first level or at the second level are numerical, they are written as described above without the use of level-indicating symbols. For example:

⠭⠂⠰⠚ means x_{1_j}

⠭⠰⠅⠂ means x_{k_1}

⠭⠰⠅⠂⠐⠬⠭⠰⠅⠆ means $x_{k_1} + x_{k_2}$

As an illustration of how numerical and [64] literal subscripts are represented, consider the following example:

The generalized associative law for addition may be stated as follows:

$$\begin{aligned}
&a_1 + a_2 + \ldots + a_n \\
&= (a_1 + a_2 + \ldots + a_{k_1}) \\
&+ (a_{k_1+1} + a_{k_1+2} + \ldots + a_{k_2}) \\
&+ (\ldots + a_{k_{m-1}}) \\
&+ (a_{k_{m-1}+1} + a_{k_{m-1}+2} \\
&+ \ldots + a_{k_m}).
\end{aligned}$$

⠁⠂⠬⠁⠆⠬ ⠠⠠⠠ ⠬⠁⠰⠝
⠨⠅ ⠪⠁⠂⠬⠁⠆⠬ ⠠⠠⠠ ⠬⠁⠰⠅⠂⠐⠕
⠬⠪⠁⠰⠅⠂⠬⠂⠐⠬⠁⠰⠅⠂⠬⠆⠐⠬ ⠠⠠⠠ ⠬⠁⠰⠅⠆⠐⠕

⠬⠪ ⠠⠠⠠ ⠬⠁⠰⠅⠰⠰⠍⠤⠂⠐⠕
⠬⠪⠁⠰⠅⠰⠰⠍⠤⠂⠰⠬⠂⠐⠬⠁⠰⠅⠰⠅⠰⠰⠍⠤⠂⠰⠬⠆
⠐⠬ ⠠⠠⠠ ⠬⠁⠰⠅⠰⠰⠍⠐⠕⠠⠲

90. Quantities Carrying Both Subscripts and Exponents — When a quantity carries both a subscript and an exponent, the subscript must be indicated first. For example:

⠭⠂⠣ means $x_1{}^2$

⠭⠂⠩⠤⠭⠆⠩ means $x_1{}^3 - x_2{}^3$

The writing of an exponent followed by a subscript implies that the subscript is carried by the exponent alone. For example: [65]

⠭⠘⠅⠂ means x^{k_1}

⠁⠂⠘⠅⠂⠐⠬⠁⠆⠘⠅⠆ means $a_1{}^{k_1} + a_2{}^{k_2}$

When it is necessary to represent a subscript which carries an exponent, the entire subscript should be enclosed within an appropriate pair of symbols of grouping. For example:

⠑⠰⠭⠐⠬⠑⠰⠪⠭⠣⠕⠐⠬⠑⠰⠪⠭⠩⠕⠐⠬⠑⠰⠪⠭⠘⠲⠰⠕

means $e_x + e_{x^2} + e_{x^3} + e_{x^4}$

⠭⠰⠁⠣ means x_a^2

⠽⠰⠁⠩⠬⠽⠰⠃⠩ means $y_a^3 + y_b^3$

XVI

SPECIFIC APPLICATIONS OF SUPERSCRIPTS AND SUBSCRIPTS

The method of representing superscripts and subscripts as described in the preceding [66] section may be applied to mathematical situations of common occurrence.

91. Representing the Limits of Integration. See Section 104, page 33, (Br. ed., page 75), for Integral Sign) — When the definite integral is represented in ink print, the interval over which the integration is performed is denoted by showing the lower end of the interval below the integral sign and showing the upper end of the interval above the integral sign. The limits of integration are then represented respectively as a subscript and a superscript. For example:

⠮⠰⠁⠘⠃⠐⠭⠣⠙⠭ means $\int_a{}^b x^2 dx$

⠮⠰⠤⠁⠘⠁⠐⠋⠪⠭⠕⠙⠭ means $\int_{-a}^{a} f(x)dx$

92. **Permutations and Combinations**—The method for writing superscripts and subscripts may be applied to the writing of permutations and combinations. For example: **[67]**

⠘⠝⠐⠠⠏⠰⠗ means $^{n}P_{r}$

⠘⠂⠆⠐⠠⠉⠢ means $^{12}C_{5}$

93. **Symbols of Comparison Used as Subscripts**—It sometimes happens that a symbol of comparison is used in a subscript, as is the case when indicating the indices over which a sum or a product is to be taken. In such cases, the symbol of comparison is neither preceded nor followed by a blank space. For example:

⠨⠠⠎⠰⠊⠨⠅⠂⠘⠝⠐⠭⠰⠊ means $\sum_{i=1}^{n} x_i$

⠨⠠⠏⠰⠅⠐⠅⠨⠅⠊⠐⠅⠨⠅⠇⠐⠭⠰⠊ means $[\Pi]_{k \leqslant i \leqslant l} x_i$

⠇⠊⠍⠰⠭⠠⠄⠴⠐⠪⠂⠬⠭⠕⠘⠂⠌⠭ means $\lim_{x \to 0} (1 + x)^{1/x}$ **[68]**

XVII

SYMBOLS AND ABBREVIATIONS USED IN GEOMETRY

94. **Symbols for "Therefore" (∴) and "Since" (∵)**—The abbreviations for "therefore" and "since" are represented in ink print by arbitrary symbols comprising three dots. In Braille, these abbreviations are represented by letter combinations suggesting these words. The abbreviation for "therefore" (∴) is ⠹⠋ (dots 1-4-5-6, 1-2-4). The abbreviation for "since" (∵) is ⠎⠉ (dots 2-3-4, 1-4).

95. **Nature of Geometric Abbreviations**—Geometric abbreviations are effected in ink print by the use of small geometric figures suggestive of the abbreviation intended. Thus, for example, the abbreviation for "is parallel to" is represented in ink print by two small parallel lines (‖); the abbreviation for "triangle" is represented by a small figure of the shape of a triangle (△).

96. **The Braille Equivalents of Geometric Abbreviations**—The geometric abbreviations in Braille are represented by the use **[69]** of letters representing the various geometric figures commonly encountered. A list of such abbreviations follows:

Line Segment ⠸⠳

Angle ⠸⠪

Right Angle ⠗⠸⠪

Triangle ⠞⠸⠪

Is Parallel To ⠏⠈⠇

Is Perpendicular To ⠏⠈⠏

Arc (of a circle or curve) ⠁⠗⠉

Circle ⠉⠈⠇

Quadrilateral ⠟⠈⠙

Rectangle ⠗⠉⠸⠪

Square ⠎⠈⠟

Parallelogram ⠏⠈⠟

97. **Capitalized Letters in Geometry and Trigonometry**—It is customary in ink print to represent geometric points by single capitalized Roman letters and to represent geometric line segments by single small Roman letters. When a mathematical discussion involves no letters other than the capitalized [70] Roman letters which represent geometric points, they may be represented in Braille by small Roman letters, thereby facilitating the reading. For example:

⠠⠇⠑⠞ ⠰⠍ ⠆ ⠮ ⠍⠊⠙⠏⠕⠔⠞ ⠷ ⠏⠟⠠⠲

Let M be the midpoint of PQ.

⠁⠕⠃ ⠯ ⠃⠕⠉ ⠜⠑ ⠎⠥⠏⠏⠇⠑⠰⠞⠜⠽
⠁⠝⠛⠇⠑⠎⠲

AOB and BOC are supplementary angles.

If, however, both small and capitalized letters occur within the same mathematical discussion, care should be taken to distinguish between them, as is the case in ink print. For example:

⠠⠇⠑⠞ ⠠⠁⠠⠃⠠⠉ ⠆ ⠁ ⠐⠗ ⠞⠗⠊⠁⠝⠛⠇⠑⠂
⠇⠑⠞ ⠠⠉ ⠙⠑⠝⠕⠞⠑ ⠮ ⠐⠗ ⠁⠝⠛⠇⠑⠂ ⠯

⠇⠑⠞ ⠰⠉ ⠙⠑⠝⠕⠞⠑ ⠮ ⠓⠽⠏⠕⠞⠢⠥⠎⠑⠲
⠠⠇⠑⠞ ⠰⠓ ⠆ ⠮ ⠁⠇⠞⠊⠞⠥⠙⠑ ⠷ ⠮
⠞⠗⠊⠁⠝⠛⠇⠑ ⠋ ⠠⠉ ⠘⠥ ⠰⠉⠂ ⠯ ⠇⠑⠞
⠰⠍ ⠯ ⠰⠝ ⠙⠑⠝⠕⠞⠑ ⠮ ⠎⠑⠛⠰⠞⠎ ⠔⠖ ⠱
⠰⠉ ⠊⠎ ⠙⠊⠧⠊⠙⠫⠲

Let ABC be a right triangle,
let C denote the right angle, and
let c denote the hypotenuse.
Let h be the altitude of the
triangle from C upon c, and let
m and n denote the segments into which
c is divided.

98. **Labeling Diagrams Accompanying the Text** — When preparing raised diagrams to accompany the Braille text, care must always be taken to label the various parts of the diagram in exact accordance with the kind of letters by which reference [71] to these parts is made in the text. If, for example, the discussion in the text concerns a triangle whose vertices have been represented in Braille by small Roman letters, then the vertices of the triangle in the diagram must be correspondingly lettered by small Roman letters. If, however, a mathematical discussion involves both small and capitalized letters, then any diagram accompanying this discussion must be correspondingly labeled by small and capitalized letters.

XVIII

SYMBOLS OF GROUPING USED FOR WRITING ADVANCED BRANCHES OF MATHEMATICS

See Sections 49-51, pages 21-22, (Br. ed., pages 46-50) for other
Symbols of Grouping.

99. **The Absolute Bars** — The absolute bars are represented by the sign ⠳ (dots 1-2-5-6) both at the beginning and at the end of the expression whose absolute value is intended. For example: [72]

⠳⠤⠆⠳ means $| - 2 |$

⠳⠭⠤⠽⠳ means $| x - y |$

⠳⠭⠳⠤⠳⠽⠳ means $| x | - | y |$

⠳⠭⠤⠽⠳⠳⠭⠬⠽⠳ means $| x - y | | x + y |$

100. Enclosing Determinants — A determinant in ink print is enclosed by a straight vertical line on each side of it. In Braille, this is represented by the sign ⠳ (dots 1-2-5-6) before the first element and after the last element of the determinant. For example:

⠳⠁⠀⠃
⠀⠉⠀⠙⠳
means $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ [73]

If it becomes necessary or desirable to distinguish between the meaning which this sign has as "absolute value" and the meaning which it has as the vertical line enclosing a determinant, the meaning which it has as "absolute value" is represented by the sign ⠳ (dots 1-2-5-6), while the meaning which it has as the vertical line enclosing a determinant is represented by the sign ⠠⠳ (dots 6, 1-2-5-6).

101. Enclosing Matrices — A matrix in ink print is enclosed by a double vertical line on each side of it. In Braille, the sign ⠳⠳ (dots 1-2-5-6, 1-2-5-6) precedes the first element and follows the last element of the matrix. For example:

⠳⠳⠁⠀⠃⠀⠉
⠀⠀⠙⠀⠑⠀⠋⠳⠳
means $\begin{Vmatrix} a & b & c \\ d & e & f \end{Vmatrix}$ [74]

XIX

OTHER MATHEMATICAL SYMBOLS USED IN ADVANCED MATHEMATICS

102. The Factorial Sign (! or $\underline{|\ \ }$) — There are two commonly used methods for representing the factorial sign in ink print. One is the square brace within which is written the quantity whose factorial is indicated. The other is the exclamation point which follows the quantity whose factorial is indicated. In Braille, both of these signs are represented by the sign ⠯ (dots 1-2-3-4-6). The appearance of this sign before a quantity implies that, in ink print, the square brace has been used. The appearance of this sign after a quantity implies that, in ink print, the exclamation point has been used. When the extent of the factorial sign is not otherwise clear, the quantity whose factorial is intended should be enclosed by an appropriate pair of symbols of grouping. For example:

⠯⠒ means $\underline{|3\ }$ (indicated in ink print by the square brace) [75]

⠝⠯ means n! (indicated in ink print by the exclamation point)

⠯⠪⠝⠤⠗⠬⠂⠕ means $\underline{|n - r + 1}$

103. Partial Derivatives — The "round D" (∂) is generally reserved in ink print for the representation of partial derivatives. In this system the ∂ is represented by the sign ⠫ (dots 1-2-4-6).

104. The Integral Sign ($\int$) See also Section 91, page 29, (Br. ed., page 66) — The integral sign is represented by the sign ⠮ (dots 2-3-4-6). For example:

⠮⠭⠣⠙⠭ means $\int x^2dx$

When, in ink print, the integral sign extends over more than a single line of writing as is the case, for example, when indicating the integral of a fraction, it is not generally necessary to indicate in Braille that the integral sign in question is larger than usual. If, however, [76] it becomes necessary or desirable to distinguish between the small and the large integral sign, then the small integral sign is represented as indicated above, while the large integral sign is represented by the sign ⠠⠮ (dots 6, 2-3-4-6).

105. The Bar (—) — The bar, in ink print, is a horizontal line above a quantity. It is represented in Braille by the sign ⠱ (dots 1-5-6).

In statistics, the bar is often used to denote the average value of a set of data. For example:

⠱⠠⠭ means $\overline{X}$ (The average value, or the mean, of X. It is often read "X bar.")

In logarithmic work, a bar above a characteristic indicates that the characteristic is negative while the mantissa which follows it is positive. For example:

⠇⠕⠛ ⠴⠨⠴⠆ ⠨⠅ ⠱⠆⠨⠒⠴⠂⠴ ⠨⠅ ⠦⠨⠒⠴⠂⠴
⠤⠂⠴⠠⠲ means log $0.02 = \bar{2}.3010 = 8.3010 - 10.$

Long expressions which are barred should be [77] enclosed within an appropriate pair of symbols of grouping to indicate the extent of the bar if this is not otherwise clear.

106. The Mathematical Asterisk (*) — The mathematical and the literary asterisk in ink print are both represented by a symbol which is star-shaped. In Braille, the asterisk is represented by the sign ⠔⠔ (dots 3-5, 3-5). It follows the quantity to which it applies without an intervening blank space. If there is any possibility of confusion between the meaning which this symbol has as 99 and the meaning which it has as a mathematical asterisk, the indicator ⠸ (dots 4-5-6) may be used to indicate that the meaning of the symbol is that of the mathematical asterisk. For example:

Let ⠭⠔⠔ denote the set of all rational numbers less than x. (The symbol x^* ⠭⠔⠔ is read "x star"). Then 1^* ⠂⠸⠔⠔ will denote the set of all rational numbers less than 1.

107. Underline (__) — If a mathematical [78] quantity is underlined in ink print, this may be represented in Braille by the sign ⠸⠱ (dots 4-5-6, 1-5-6). If the extent of the underline is not otherwise clear, the quantity concerned should be enclosed within an appropriate pair of symbols of grouping.

108. Time Derivatives — It is common usage in ink print to represent the time derivative of a quantity by placing a dot above it. In Braille, this is represented by the sign ⠈ (dot 4) immediately following the quantity concerned. For example:

⠭⠈ ⠨⠅ ⠙⠭⠌⠙⠞ is read: $\dot{x} = \frac{dx}{dt}$

Similarly, the second time derivative of x may be represented by ⠭⠈⠈ when the double dot is used in ink print for this purpose.

109. "Del" (∇) — When "del" is represented in ink print by the inverted delta, it is represented in Braille by the sign ⠸⠫ (dots 4-5-6, 1-2-4-6). [79]

XX

THE REPRESENTATION OF FUNCTIONS HAVING SPECIAL NAMES

Some mathematical functions have been assigned special names and standard abbreviations have been provided for them.

110. Logarithms — The abbreviation for "logarithm" is **log**. The abbreviation for "cologarithm" is **colog**. The abbreviation for "antilogarithm" is **antilog**. These abbreviations are represented in Braille in conformity with the ink-print usage.

The abbreviation for "natural logarithm" is sometimes used in ink print and is **ln**. When this usage is adopted in ink print, it is also adopted in Braille.

111. The Trigonometric Functions — The trigonometric functions are as follows: [80]

sine	sin	⠎⠊⠝	cosine	cos	⠉⠕⠎
tangent	tan	⠞⠁⠝	cotangent	cot	⠉⠕⠞
secant	sec	⠎⠑⠉	cosecant	csc	⠉⠎⠉

Notice that the "in" contraction may not be used in writing the abbreviation for "sine."

112. **The Inverse Trigonometric Functions** — There are two general methods employed in ink print for the designation of the inverse trigonometric functions.

When the ink print employs the abbreviation "arc" to precede the trigonometric function, the Braille text does likewise, except that no space is left between the abbreviation "arc" and the abbreviation for the trigonometric function which follows it. The "ar" sign may not be used in the abbreviation for "arc." For example:

⠁⠗⠉⠎⠊⠝ ⠂⠌⠆ ⠨⠅ ⠨⠏⠌⠖ means $\text{arc sin } \frac{1}{2} = \frac{\pi}{6}$

When, in ink print, the exponent $^{-1}$ is used to denote the inverse function, the same procedure is followed in the Braille [81] text. For example:

⠎⠊⠝⠘⠤⠂ ⠜⠒⠌⠆ ⠨⠅ ⠨⠏⠌⠒ means $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

113. **The Hyperbolic Functions** — The abbreviations for the hyperbolic functions are as follows:

hyperbolic sine	sinh	⠎⠊⠝⠓	hyperbolic cosine	cosh	⠉⠕⠎⠓
hyperbolic tangent	tanh	⠞⠁⠝⠓	hyperbolic cotangent	coth	⠉⠕⠞⠓
hyperbolic secant	sech	⠎⠑⠉⠓	hyperbolic cosecant	csch	⠉⠎⠉⠓

The contractions for "ch," "sh" and "th" may not be used in writing these abbreviations.

114. **The Inverse Hyperbolic Functions** — The inverse hyperbolic functions are treated in the same manner as the inverse trigonometric functions, in accordance with the two general methods employed in ink print for [82] this purpose. The Braille text should conform to the ink-print usage.

115. Del — When, in ink print, the abbreviation "del" is employed instead of the symbol for this function, the same practice is followed in the Braille text.

116. Limit — The abbreviation for "limit" is **lim.**

117. **Rules for the Writing of Special Functions** — All of the special functions listed in this section should, in general, be followed by a blank space. For example:

⠇⠕⠛ ⠭ means $\log x$
⠎⠊⠝ ⠠⠁ means $\sin A$
⠇⠊⠍ ⠨⠋ means $\lim \phi$

When one of these abbreviations carries an exponent, no space is left between the abbreviation and the exponent, but a space is left after the exponent, which thereby terminates its effect. For example:

⠇⠕⠛⠣ ⠭ means $\log^2 x$ [83]

⠉⠕⠎⠣ ⠨⠹ means $\cos^2 \theta$

⠞⠁⠝⠘⠤⠂ ⠜⠒ means $\tan^{-1}\sqrt{3}$

When one of these abbreviations carries a subscript, no space is left between the abbreviation and the subscript, but a space is left after the subscript which thereby terminates its effect. For example:

⠇⠕⠛⠂⠴ ⠆ ⠨⠅ ⠴⠨⠒⠴⠂⠴⠒⠠⠲ means $\text{Log}_{10}\ 2 = 0.30103$.

⠇⠕⠛⠰⠑ ⠆⠨⠶⠂⠦⠒ ⠨⠡⠅ ⠂⠠⠲ means $\text{Log}_e\ 2.7183 \approx 1$.

When one of these abbreviations is preceded by a coefficient or by a sign of operation, no space is left before the abbreviation. For example:

⠎⠊⠝ ⠆⠭ ⠨⠅ ⠆⠎⠊⠝ ⠭ ⠉⠕⠎ ⠭⠠⠲

means $\sin 2x = 2 \sin x \cos x$.

⠎⠊⠝⠣ ⠭ ⠬⠉⠕⠎⠣ ⠭ ⠨⠅ ⠂⠠⠲

means $\sin^2 x + \cos^2 x = 1$.

⠭ ⠨⠅ ⠭⠄⠉⠕⠎ ⠨⠹ ⠤⠽⠄⠎⠊⠝ ⠨⠹⠠⠲

means $x = x' \cos \theta - y' \sin\ 0$.

When one of these abbreviations is itself an exponent, it is preceded by the appropriate [84] level-indicating symbol without a space but is followed by a space. In this case, however, any space which follows one of these abbreviations does not terminate its exponential effect. For example:

⠑⠘⠎⠊⠝ ⠭ means $e^{\sin x}$

⠑⠘⠎⠊⠝ ⠭ ⠬⠉⠕⠎ ⠭ means $e^{\sin x + \cos x}$

⠑⠘⠎⠊⠝ ⠭ ⠐⠬⠉⠕⠎ ⠭ means $e^{\sin x} + \cos x$

However, a symbol of comparison terminates the effect of a level-indicating symbol used before one of these abbreviations. For example:

⠑⠘⠎⠊⠝ ⠭ ⠬⠉⠕⠎ ⠭ ⠨⠅ ⠽ means $e^{\sin x + \cos x} = y$

XXI

THE DIVISION OF MATHEMATICAL EXPRESSIONS

The division of a mathematical expression [85] at the end of a line should be avoided if possible.

118. The First Choice — When it is necessary to divide a mathematical expression at the end of a line, the division should be made, if possible, before a symbol of comparison. The symbol of comparison then stands as the first symbol on the following line.

119. The Second Choice — If it is not possible to divide a mathematical expression before a symbol of comparison, the division should be made before a plus sign or a minus sign. Such signs then stand as the first symbol at the beginning of the following line.

120. The Third Choice — If it is not possible to make the division in accordance with the first two choices, the third choice is to divide the expression between factors which are enclosed within symbols of grouping. It may then become desirable to prefix the opening symbol of grouping which appears on the following line by the multiplication sign.

No other division may be made unless a factor is so long as not to be contained on a [86] single line. In that case, division should be made between the terms of that factor.

Note: It follows from the above rules that whenever a division is made, a symbol of comparison, a plus sign or a minus sign, or a multiplication sign stands as the first symbol of the following line and serves as an indication that a mathematical expression has been divided.

When it is necessary to divide a mathematical expression, it is preferable that the margin of the line on which the expression is continued be two cells wider or two cells narrower than the margin of the line on which the expression begins.

XXII

ILLUSTRATIVE EXAMPLES IN ARITHMETIC

121. A Multiplication Example:

Multiply 926 by 345. [87] Solution:

⠔⠆⠖	926
⠒⠲⠢	345
⠒⠒⠒⠒⠒⠒	———
⠲⠖⠒⠴	4630
⠒⠶⠴⠲	3704
⠆⠶⠶⠦	2778
⠒⠒⠒⠒⠒⠒	———
⠒⠂⠔⠲⠶⠴	319470

122. A Division Example:

Divide 379 into 51,633. Solution:

⠒⠶⠔⠕⠢⠂⠖⠒⠒⠪⠂⠒⠖⠀⠦⠔
⠀⠀⠀⠀⠒⠶⠔
⠀⠀⠀⠀⠂⠒⠶⠒
⠀⠀⠀⠀⠂⠂⠒⠶
⠀⠀⠀⠀⠀⠆⠒⠖⠒
⠀⠀⠀⠀⠀⠆⠆⠶⠲
⠀⠀⠀⠀⠀⠀⠀⠦⠔

379)$\overline{51,633}$(136
379
1373
1137
2363
2274
89

[88]

123. An Example Involving Fractions:

Simplify:

$$\frac{\frac{3/4}{2/5}}{\frac{2/3}{5/6}}$$

⠠⠹⠹⠒⠌⠲⠼⠌⠹⠆⠌⠢⠼⠌⠌⠹⠆⠌⠒⠼⠌⠹⠢⠌⠖⠼⠠⠼⠠⠲

Solution:

$$\frac{\frac{3/4}{2/5}}{\frac{2/3}{5/6}} = \frac{3/4 \times 5/2}{2/3 \times 6/5} = \frac{15/8}{12/15} = 15/8 \times 15/12 = 225/96 = 75/32 = 2\text{-}11/32$$

⠠⠹⠹⠒⠌⠲⠼⠌⠹⠆⠌⠢⠼⠌⠌⠹⠆⠌⠒⠼⠌⠹⠢⠌⠖⠼⠠⠼
⠀⠀⠨⠅⠀⠀⠠⠹⠹⠒⠌⠲⠼⠹⠢⠌⠆⠼⠌⠌⠹⠆⠌⠒⠼⠹⠖⠌⠢⠼⠠⠼
⠀⠀⠨⠅⠀⠀⠠⠹⠹⠂⠢⠌⠦⠼⠌⠌⠹⠂⠆⠌⠂⠢⠼⠠⠼
⠀⠀⠨⠅⠀⠀⠹⠂⠢⠌⠦⠼⠹⠂⠢⠌⠂⠆⠼⠀⠨⠅⠀⠆⠆⠢⠌⠔⠖
⠀⠀⠨⠅⠀⠀⠶⠢⠌⠒⠆⠀⠨⠅⠀⠆⠸⠹⠂⠂⠌⠒⠆⠠⠲

[89]

XXIII

ILLUSTRATIVE EXAMPLES IN ALGEBRA

124. An Equation with Fractional Coefficients:

Solve and check the equation $\frac{5}{4}x - \frac{3}{2} = 0.$ ⠢⠌⠲⠼⠭⠤⠒⠌⠆ ⠨⠅ ⠴⠠⠲

$\frac{5}{4}x - \frac{3}{2} = 0$ ⠢⠌⠲⠼⠭⠤⠒⠌⠆ ⠨⠅ ⠴
$\frac{5}{4}x = \frac{3}{2}$ ⠢⠌⠲⠼⠭ ⠨⠅ ⠒⠌⠆
$5x = 6$ ⠢⠭ ⠨⠅ ⠖
$x = \frac{6}{5}.$ ⠭ ⠨⠅ ⠖⠌⠢⠠⠲

Check:

$\frac{5}{4}\left(\frac{6}{5}\right) - \frac{3}{2} = 0?$ ⠢⠌⠲⠼⠖⠌⠢⠤⠒⠌⠆ ⠨⠅ ⠴⠠⠦
$\frac{30}{20} - \frac{3}{2} = 0?$ ⠒⠴⠌⠆⠴⠤⠒⠌⠆ ⠨⠅ ⠴⠠⠦
$\frac{3}{2} - \frac{3}{2} = 0?$ ⠒⠌⠆⠤⠒⠌⠆ ⠨⠅ ⠴⠠⠦
$0 = 0?$ ⠴ ⠨⠅ ⠴⠠⠦ Yes.

125. A System of Two Simultaneous Linear Equations:

Find the solution for the system of equations

$x + 3y = 4,$
⠭⠬⠒⠽ ⠨⠅ ⠲⠠⠂

$2x + y = 3$
⠆⠭⠬⠽ ⠨⠅ ⠒

and check.

Solution:

$x + 3y = 4,$
⠭⠬⠒⠽ ⠨⠅ ⠲⠠⠂
$2x + 6y = 8,$
⠆⠭⠬⠖⠽ ⠨⠅ ⠦⠠⠂
$5y = 5,$
⠢⠽ ⠨⠅ ⠢⠠⠂

$2x + y = 3$
⠆⠭⠬⠽ ⠨⠅ ⠒
$2x + y = 3$
⠆⠭⠬⠽ ⠨⠅ ⠒

$y = 1.$

⠽ ⠨⠅ ⠂⠠⠲ [90]

Substitute $y = 1$ in the first equation. Then $x + 3 \times 1 = 4$ ⠭⠬⠒⠡⠂ ⠨⠅ ⠲

$x + 3 = 4$

⠭⠬⠒ ⠨⠅ ⠲

$x = 1.$

⠭ ⠨⠅ ⠂⠠⠲

Check:

For the first equation:

$1 + 3 \times 1 = 4?$

⠂⠬⠒⠡⠂ ⠨⠅ ⠲⠠⠦

$1 + 3 = 4?$

⠂⠬⠒ ⠨⠅ ⠲⠠⠦

$4 = 4?$

⠲ ⠨⠅ ⠲⠠⠦ Yes.

For the second equation:

$2 \times 1 + 1 = 3?$

⠆⠡⠂⠬⠂ ⠨⠅ ⠒⠠⠦

$2 + 1 = 3?$

⠆⠬⠂ ⠨⠅ ⠒⠠⠦

$3 = 3?$

⠒ ⠨⠅ ⠒⠠⠦ Yes.

126. **A Quadratic Equation:**

Solve for the quadratic equation $2x^2 - 7x + 3 = 0$ and check.

⠆⠭⠣⠤⠶⠭⠬⠒ ⠨⠅ ⠴

Solution:

The formula for the solution of a quadratic equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac.}}{2a}$$

⠭ ⠨⠅ ⠹⠤⠃⠬⠤⠜⠃⠣⠤⠲⠁⠉⠻⠌⠆⠁⠼⠠⠲

We have

$a = 2,$ $b = -7,$ $c = 3$

⠁ ⠨⠅ ⠆⠠⠂ ⠃ ⠨⠅ ⠤⠶⠠⠂ ⠉ ⠨⠅ ⠒

$$x = \frac{7 \pm \sqrt{49 - 4 \times 2 \times 3}}{2 \times 2}$$

⠭ ⠨⠅ ⠹⠶⠬⠤⠜⠲⠔⠤⠲⠡⠆⠡⠒⠻⠌⠆⠡⠆⠼

$$= \frac{7 \pm \sqrt{49 - 24}}{4}$$

⠨⠅ ⠹⠶⠬⠤⠜⠲⠔⠤⠆⠲⠻⠌⠲⠼

$$= \frac{7 \pm \sqrt{25}}{4} = \frac{7 \pm 5}{4}$$

⠨⠅ ⠹⠶⠬⠤⠜⠆⠢⠌⠲⠼ ⠨⠅ ⠹⠶⠬⠤⠢⠌⠲⠼

$$x = \frac{7 + 5}{4} = 12/4 = 3$$

⠭ ⠨⠅ ⠹⠶⠬⠢⠌⠲⠼ ⠨⠅ ⠂⠆⠌⠲ ⠨⠅ ⠒

$$x = \frac{7 - 5}{4} = 2/4 = 1/2$$

⠭ ⠨⠅ ⠹⠶⠤⠢⠌⠲⠼ ⠨⠅ ⠆⠌⠲ ⠨⠅ ⠂⠌⠆

Check:

Consider the root $x = 3.$ ⠭ ⠨⠅ ⠒⠠⠲ [91]

$2 \times 3^2 - 7 \times 3 + 3 = 0?$

⠆⠡⠒⠣⠤⠶⠡⠒⠬⠒ ⠨⠅ ⠴⠠⠦

$2 \times 9 - 7 \times 3 + 3 = 0?$

⠆⠡⠔⠤⠶⠡⠒⠬⠒ ⠨⠅ ⠴⠠⠦

$18 - 21 + 3 = 0?$

⠂⠦⠤⠆⠂⠬⠒ ⠨⠅ ⠴⠠⠦

$21 - 21 = 0?$

⠆⠂⠤⠆⠂ ⠨⠅ ⠴⠠⠦

$0 = 0?$ Yes.

⠴ ⠨⠅ ⠴⠠⠦

Consider the root $x = 1/2$ ⠭ ⠨⠅ ⠂⠌⠆

$2(1/2)^2 - 7(1/2) + 3 = 0?$

⠆⠪⠂⠌⠆⠕⠣⠤⠶⠹⠂⠌⠆⠼⠬⠒ ⠨⠅ ⠴⠠⠦

$2(1/4) - 7(1/2) + 3 = 0?$

⠆⠹⠂⠌⠲⠼⠤⠶⠹⠂⠌⠆⠼⠬⠒ ⠨⠅ ⠴⠠⠦

$1/2 - 7/2 + 3 = 0?$

⠂⠌⠆⠤⠶⠌⠆⠬⠒ ⠨⠅ ⠴⠠⠦

$- 6/2 + 3 = 0?$

⠤⠖⠌⠆⠬⠒ ⠨⠅ ⠴⠠⠦

$- 3 + 3 = 0?$

⠤⠒⠬⠒ ⠨⠅ ⠴⠠⠦

$0 = 0?$ Yes.

⠴ ⠨⠅ ⠴⠠⠦

127. The Sum of the Terms of a Geometric Progression:

Derive the formula for the sum of the first n terms of a geometric progression.

Derivation:

Let S_n denote the sum of the first n terms of a geometric progression, let a denote the first term of the progression, and let r denote the ratio by which the progression is formed. Then

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

⠠⠎⠰⠝ ⠨⠅ ⠁⠬⠁⠗⠬⠁⠗⠣⠬ ⠠⠠⠠ ⠬⠁⠗⠘⠝⠤⠂

Multiply each member of the above equation by r. Then

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$

⠗⠠⠎⠰⠝ ⠨⠅ ⠁⠗⠬⠁⠗⠣⠬⠁⠗⠩⠬ ⠠⠠⠠
⠬⠁⠗⠘⠝

Subtract the second equation from the first. Then

$S_n - rS_n = a - ar^n$

⠠⠎⠰⠝⠐⠤⠗⠠⠎⠰⠝ ⠨⠅ ⠁⠤⠁⠗⠘⠝ [92]

Write both members as the product of their factors. Then

$S_n\ (1 - r) = a\ (1 - r^n)$

⠠⠎⠰⠝⠐⠪⠂⠤⠗⠕ ⠨⠅ ⠁⠪⠂⠤⠗⠘⠝⠐⠕

If $r \neq 1$ ⠗ ⠨⠌⠅ ⠂, we may divide both members by $1 - r$ ⠂⠤⠗. Then

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

⠠⠎⠰⠝ ⠨⠅ ⠹⠁⠪⠂⠤⠗⠘⠝⠐⠕⠌⠂⠤⠗⠼

This is the required formula.

128. Cramer's Rule:

Use Cramer's rule in finding the solution for the system of three simultaneous linear equations.

$x + 2y - z = 0$

⠭⠬⠆⠽⠤⠵ ⠨⠅ ⠴

$x + y + 2z = 1$

⠭⠬⠽⠬⠆⠵ ⠨⠅ ⠂

$4x - y + z = 2$

⠲⠭⠤⠽⠬⠵ ⠨⠅ ⠆

and check.

Solution: Let

$$a = \begin{matrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{matrix}$$

⠁ ⠨⠅ ⠳⠴ ⠆ ⠤⠂
⠂ ⠂ ⠆
⠆ ⠤⠂ ⠂⠳

$$b = \begin{matrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 4 & 2 & 1 \end{matrix}$$

⠃ ⠨⠅ ⠳⠂ ⠴ ⠤⠂
⠂ ⠂ ⠆
⠲ ⠆ ⠂⠳

$$c = \begin{matrix}1 & 2 & 0\\ 1 & 1 & 1\\ 4 & -1 & 2\end{matrix}$$

⠉ ⠨⠅ ⠳⠂ ⠆ ⠴
⠂ ⠂ ⠂
⠲ ⠤⠂ ⠆⠳

$$d = \begin{matrix}1 & 2 & -1\\ 1 & 1 & 2\\ 4 & -1 & 1\end{matrix}$$

⠙ ⠨⠅ ⠳⠂ ⠆ ⠤⠂
⠂ ⠂ ⠆
⠲ ⠤⠂ ⠂⠳

a $= -[(2 \times 1) - (-1 \times -1)]$
⠁ ⠨⠅ ⠤⠷⠪⠆⠡⠂⠕⠤⠪⠤⠂⠡⠤⠂⠕⠾

$+ 2[(2 \times 2) - (1 \times -1)]$
⠬⠆⠷⠪⠆⠡⠆⠕⠤⠪⠂⠡⠤⠂⠕⠾

$= -(2 - 1) + 2(4 + 1) = -1 + 2 \times 5$
⠨⠅ ⠤⠪⠆⠤⠂⠕⠬⠆⠪⠲⠬⠂⠕ ⠨⠅ ⠤⠂⠬⠆⠡⠢

$= -1 + 10 = 9.$
⠨⠅ ⠤⠂⠬⠂⠴ ⠨⠅ ⠔⠠⠲ [93]

b $= [(1 \times 1) - (2 \times 2)] - [(1 \times 2)$
⠃ ⠨⠅ ⠷⠪⠂⠡⠂⠕⠤⠪⠆⠡⠆⠕⠾⠤⠷⠪⠂⠡⠆⠕

$- (4 \times 1)] = (1 - 4) - (2 - 4)$
⠤⠪⠲⠡⠂⠕⠾ ⠨⠅ ⠪⠂⠤⠲⠕⠤⠪⠆⠤⠲⠕

$= -3 + 2 = -1.$
⠨⠅ ⠤⠒⠬⠆ ⠨⠅ ⠤⠂⠠⠲

c = [(1 × 2) — (— 1 × 1)] — 2 [(1 × 2)
⠉ ⠨⠅ ⠷⠪⠂⠡⠆⠕⠤⠪⠤⠂⠡⠂⠕⠾⠤⠆⠷⠪⠂⠡⠆⠕

— (4 × 1)] = (2 + 1) — 2 (2 — 4)
⠤⠪⠲⠡⠂⠕⠾ ⠨⠅ ⠪⠆⠬⠂⠕⠤⠆⠪⠆⠤⠲⠕

= 3 — 2 × — 2 = 3 + 4 = 7.
⠨⠅ ⠒⠤⠆⠡⠤⠆ ⠨⠅ ⠒⠬⠲ ⠨⠅ ⠶⠠⠲

d = [(1 × 1) — (— 1 × 2)] — [(2 × 1)
⠙ ⠨⠅ ⠷⠪⠂⠡⠂⠕⠤⠪⠤⠂⠡⠆⠕⠾⠤⠷⠪⠆⠡⠂⠕

— (— 1 × —1)] + 4 [(2 × 2) — (1 × — 1)]
⠤⠪⠤⠂⠡⠤⠂⠕⠾⠬⠲⠷⠪⠆⠡⠆⠕⠤⠪⠂⠡⠤⠂⠕⠾

= (1 + 2) — (2 — 1) + 4(4 + 1)
⠨⠅ ⠪⠂⠬⠆⠕⠤⠪⠆⠤⠂⠕⠬⠲⠪⠲⠬⠂⠕

= 3 — 1 + 4 × 5 = 3 — 1 + 20
⠨⠅ ⠒⠤⠂⠬⠲⠡⠢ ⠨⠅ ⠒⠤⠂⠬⠆⠴

= 23 — 1 = 22.
⠨⠅ ⠆⠒⠤⠂ ⠨⠅ ⠆⠆⠠⠲

Apply Cramer's rule. Then

x = a/d = 9/22,
⠭ ⠨⠅ ⠁⠌⠙ ⠨⠅ ⠔⠌⠆⠆⠠⠂

y = b/d = — 1/22,
⠽ ⠨⠅ ⠃⠌⠙ ⠨⠅ ⠤⠂⠌⠆⠆⠠⠂

z = c/d = 7/22.
⠵ ⠨⠅ ⠉⠌⠙ ⠨⠅ ⠶⠌⠆⠆⠠⠲

Check. For the first equation:

9/22 + 2(— 1/22) — 7/22 = 0?
⠔⠌⠆⠆⠬⠆⠹⠤⠂⠌⠆⠆⠼⠤⠶⠌⠆⠆ ⠨⠅ ⠴⠠⠦

9/22 — 2/22 — 7/22 = 0?

⠔⠌⠆⠆⠤⠆⠌⠆⠆⠤⠶⠌⠆⠆ ⠨⠅ ⠴⠠⠦

9/22 — 9/22 = 0?

⠔⠌⠆⠆⠤⠔⠌⠆⠆ ⠨⠅ ⠴⠠⠦

0 = 0? Yes.

⠴ ⠨⠅ ⠴⠠⠦

For the second equation:

9/22 — 1/22 + 2(7/22) = 1?

⠔⠌⠆⠆⠤⠂⠌⠆⠆⠬⠆⠹⠶⠌⠆⠆ ⠨⠅ ⠂⠠⠦

9/22 — 1/22 + 14/22 = 1?

⠔⠌⠆⠆⠤⠂⠌⠆⠆⠬⠂⠲⠌⠆⠆ ⠨⠅ ⠂⠠⠦

23/22 — 1/22 = 1?

⠆⠒⠌⠆⠆⠤⠂⠌⠆⠆ ⠨⠅ ⠂⠠⠦

22/22 = 1?

⠆⠆⠌⠆⠆ ⠨⠅ ⠂⠠⠦

1 = 1? Yes.

⠂ ⠨⠅ ⠂⠠⠦ [94]

For the third equation:

4(9/22) + 1/22 + 7/22 = 2?

⠲⠹⠔⠌⠆⠆⠼⠬⠂⠌⠆⠆⠬⠶⠌⠆⠆ ⠨⠅ ⠆⠠⠦

36/22 + 1/22 + 7/22 = 2?

⠒⠖⠌⠆⠆⠬⠂⠌⠆⠆⠬⠶⠌⠆⠆ ⠨⠅ ⠆⠠⠦

44/22 = 2?

⠲⠲⠌⠆⠆ ⠨⠅ ⠆⠠⠦

2 = 2? Yes.

⠆ ⠨⠅ ⠆⠠⠦

XXIV

AN ILLUSTRATION FROM GEOMETRY

129. **The Pythagorean Theorem:**

Show that, for every right triangle, it is necessary that the square of the hypotenuse shall equal the sum of the squares of the remaining two sides.

Proof:

Before proving the above theorem, we shall first prove a lemma.

Lemma: If an altitude be constructed upon the hypotenuse of any right triangle, then the segments into which the hypotenuse is thereby divided are such that each leg of the right triangle is the mean proportional between the segment to which it is adjacent and the whole hypotenuse.

Proof of the lemma:

Let ABC ⠠⠁⠠⠃⠠⠉ be any right triangle with C ⠰⠠⠉ as the right angle. Denote the sides of the triangle which are opposite these angles by a, ⠰⠁ b, ⠰⠃ and c, ⠰⠉ respectively. Let **h** ⠰⠓ denote the [95] altitude upon **c** ⠰⠉ from C, ⠰⠠⠉ and let **P** ⠰⠠⠏ denote the foot of this altitude. Let **m** ⠰⠍ denote the segment AP, ⠠⠁⠠⠏ and let **n** ⠰⠝ denote the segment PB ⠠⠏⠠⠃. We must show that **b** ⠰⠃ is the mean proportional between **m** ⠰⠍ and **c**, ⠰⠉ and that a ⠰⠁ is the mean proportional between **n** ⠰⠝ and **c** ⠰⠉.

Consider the two right triangles △ ABC ⠞⠸⠪ ⠠⠁⠠⠃⠠⠉ and △ ACP ⠞⠸⠪ ⠠⠁⠠⠉⠠⠏. In these triangles, ∠A = ∠A ⠸⠪ ⠠⠁ ⠨⠅ ⠸⠪ ⠠⠁ by identity, and ∠P = ∠C ⠸⠪ ⠠⠏ ⠨⠅ ⠸⠪ ⠠⠉ since these are both right angles and all right angles are equal. ∴ △ ACP ⠹⠋ ⠞⠸⠪ ⠠⠁⠠⠉⠠⠏ is similar to △ ABC ⠞⠸⠪ ⠠⠁⠠⠃⠠⠉ since two triangles are similar if two angles of one are equal respectively to two angles of the other. ∴ $\mathbf{m}/\mathbf{b} = \mathbf{b}/\mathbf{c}$

⠹⠋ ⠍⠌⠃ ⠨⠅ ⠃⠌⠉ since corresponding parts of similar triangles stand in the same ratio to each other. This shows that **b** ⠰⠃ is the mean proportional between **m** ⠰⠍ and **c** ⠰⠉.

By considering the right triangles △ ABC ⠞⠸⠪ ⠠⠁⠠⠃⠠⠉ and △ CBP, ⠞⠸⠪ ⠠⠉⠠⠃⠠⠏ we see that **a** ⠰⠁ is the mean proportional between **n** ⠰⠝ and **c** ⠰⠉ by the same reasoning. Thus, our lemma is **[96]** proved.

Proof of the Pythagorean Theorem:

We use the same notation. Since **b** ⠰⠃ is the mean proportional between **m** ⠰⠍ and **c** ⠰⠉, we may write $\mathbf{b}^2 = \mathbf{mc}$ ⠃⠣ ⠨⠅ ⠍⠉. Since **a** ⠰⠁ is the mean proportional between **n** ⠰⠝ and **c** ⠰⠉, we may write $\mathbf{a}^2 = \mathbf{nc}$ ⠁⠣ ⠨⠅ ⠝⠉. Adding these two equations, we have $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{mc} + \mathbf{nc}$ ⠁⠣⠬⠃⠣ ⠨⠅ ⠍⠉⠬⠝⠉. If we factor the right member, we have $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}(\mathbf{m} + \mathbf{n})$ ⠁⠣⠬⠃⠣ ⠨⠅ ⠉⠪⠍⠬⠝⠕. But $\mathbf{m} + \mathbf{n} = \mathbf{c}$, ⠍⠬⠝ ⠨⠅ ⠉ so that $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c} \times \mathbf{c}$ ⠁⠣⠬⠃⠣ ⠨⠅ ⠉⠡⠉, or $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$. ⠁⠣⠬⠃⠣ ⠨⠅ ⠉⠣

This proves the theorem.

XXV

ILLUSTRATIVE EXAMPLES IN TRIGONOMETRY

130. A Trigonometric Identity:

Show that if $a + b + c = 180°$ ⠁⠬⠃⠬⠉ ⠨⠅ ⠂⠦⠴⠙⠛

then $\tan a + \tan b + \tan c = \tan a \tan b \tan c$.

⠞⠁⠝ ⠁ ⠬⠞⠁⠝ ⠃ ⠬⠞⠁⠝ ⠉
⠨⠅ ⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃ ⠞⠁⠝ ⠉⠠⠲

Proof:

$$\tan A + \tan B + \tan C = \tan A + \tan B + \tan (180^\circ - [A + B])$$

⠞⠁⠝ ⠁ ⠬⠞⠁⠝ ⠃ ⠬⠞⠁⠝ ⠉ ⠨⠅ ⠞⠁⠝ ⠁
⠬⠞⠁⠝ ⠃ ⠬⠞⠁⠝ ⠪⠂⠦⠴⠙⠛ ⠤⠷⠁⠬⠃⠾⠕

$$= \tan A + \tan B - \tan (A + B)$$

⠨⠅ ⠞⠁⠝ ⠁ ⠬⠞⠁⠝ ⠃ ⠤⠞⠁⠝ ⠪⠁⠬⠃⠕

$$= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

⠨⠅ ⠞⠁⠝ ⠁ ⠬⠞⠁⠝ ⠃ ⠤⠹⠞⠁⠝ ⠁
⠬⠞⠁⠝ ⠃ [97] ⠌⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠼

$$= \frac{\tan A\ (1 - \tan A \tan B) + \tan B\ (1 - \tan A \tan B) - \tan A - \tan B}{1 - \tan A \tan B}$$

⠨⠅ ⠹⠞⠁⠝ ⠁ ⠪⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠕
⠬⠞⠁⠝ ⠃ ⠪⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠕
⠤⠞⠁⠝ ⠁ ⠤⠞⠁⠝ ⠃
⠌⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠼

$$= \frac{\tan A - \tan^2 A \tan B + \tan B - \tan A \tan^2 B - \tan A - \tan B}{1 - \tan A \tan B}$$

⠨⠅ ⠹⠞⠁⠝ ⠁ ⠤⠞⠁⠝⠣ ⠁ ⠞⠁⠝ ⠃ ⠬⠞⠁⠝ ⠃
⠤⠞⠁⠝ ⠁ ⠞⠁⠝⠣ ⠃ ⠤⠞⠁⠝ ⠁ ⠤⠞⠁⠝ ⠃
⠌⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠼

$$= \frac{- \tan^2 A \tan B - \tan A \tan^2 B}{1 - \tan A \tan B}$$

⠨⠅ ⠹⠤⠞⠁⠝⠣ ⠁ ⠞⠁⠝ ⠃
⠤⠞⠁⠝ ⠁ ⠞⠁⠝⠣ ⠃
⠌⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠼

$$= \frac{-\tan A - \tan B}{1 - \tan A \tan B} \times \tan A \tan B$$

⠨⠅ ⠹⠤⠞⠁⠝ ⠁ ⠤⠞⠁⠝ ⠃
⠌⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠼
⠡⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃

$$= -\frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan A \tan B$$

⠨⠅ ⠤⠹⠞⠁⠝ ⠁ ⠬⠞⠁⠝ ⠃
⠌⠂⠤⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃⠼
⠡⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃

$$= -\tan (A + B) \tan A \tan B$$

⠨⠅ ⠤⠞⠁⠝ ⠪⠁⠬⠃⠕ ⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃

$$= \tan (180^\circ - [A + B]) \tan A \tan B$$

⠨⠅ ⠞⠁⠝ ⠪⠂⠦⠴⠙⠛ ⠤⠷⠁⠬⠃⠾⠕
⠡⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃

$$= \tan C \tan A \tan B$$

⠨⠅ ⠞⠁⠝ ⠉ ⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃

$$= \tan A \tan B \tan C$$

⠨⠅ ⠞⠁⠝ ⠁ ⠞⠁⠝ ⠃ ⠞⠁⠝ ⠉

131. A Trigonometric Equation:

Find the values of theta θ which satisfy the equation

$$2 \cos^2 2\,\theta + 14 \sin \theta \cos \theta - 5 = 0$$

⠆⠉⠕⠎⠣ ⠆⠨⠹ ⠬⠂⠲⠎⠊⠝ ⠨⠹ ⠉⠕⠎ ⠨⠹ ⠤⠢
⠨⠅ ⠴

and which lie between 0° ⠴⠙⠛ and 360° ⠒⠖⠴⠙⠛.

Solution:

$2 \cos^2 2\,\theta + 14 \sin \theta \cos \theta - 5 = 0$

⠆⠉⠕⠎⠣ ⠆⠨⠹ ⠬⠂⠲⠎⠊⠝ ⠨⠹ ⠉⠕⠎ ⠨⠹ ⠤⠢
⠨⠅ ⠴ [98]

$2 \cos^2 2\,\theta + 7 \sin 2\,\theta - 5 = 0$

⠆⠉⠕⠎⠣ ⠆⠨⠹ ⠬⠶⠎⠊⠝ ⠆⠨⠹ ⠤⠢ ⠨⠅ ⠴

$2(1 - \sin^2 2\,\theta) + 7 \sin 2\,\theta - 5 = 0$

⠆⠪⠂⠤⠎⠊⠝⠣ ⠆⠨⠹⠕ ⠬⠶⠎⠊⠝ ⠆⠨⠹ ⠤⠢
⠨⠅ ⠴

$2 - 2 \sin^2 2\,\theta + 7 \sin 2\,\theta - 5 = 0$

⠆⠤⠆⠎⠊⠝⠣ ⠆⠨⠹ ⠬⠶⠎⠊⠝ ⠆⠨⠹ ⠤⠢ ⠨⠅ ⠴

$- 2 \sin^2 2\,\theta + 7 \sin 2\,\theta - 3 = 0$

⠤⠆⠎⠊⠝⠣ ⠆⠨⠹ ⠬⠶⠎⠊⠝ ⠆⠨⠹ ⠤⠒ ⠨⠅ ⠴

$2 \sin^2 2\,\theta - 7 \sin 2\,\theta + 3 = 0$

⠆⠎⠊⠝⠣ ⠆⠨⠹ ⠤⠶⠎⠊⠝ ⠆⠨⠹ ⠬⠒ ⠨⠅ ⠴

$(\sin 2\,\theta - 3)(2 \sin 2\,\theta - 1) = 0$

⠪⠎⠊⠝ ⠆⠨⠹ ⠤⠒⠕⠪⠆⠎⠊⠝ ⠆⠨⠹ ⠤⠂⠕ ⠨⠅ ⠴

If $\sin 2\,\theta - 3 = 0$, ⠎⠊⠝ ⠆⠨⠹ ⠤⠒ ⠨⠅ ⠴⠠⠂ then $\sin 2\,\theta = 3$, ⠎⠊⠝ ⠆⠨⠹ ⠨⠅ ⠒⠠⠂ and there exists no real angle for which this is true so that this factor yields no solution. We must then consider

$2 \sin 2\,\theta - 1 = 0$

⠆⠎⠊⠝ ⠆⠨⠹ ⠤⠂ ⠨⠅ ⠴

$2 \sin 2\,\theta = 1$

⠆⠎⠊⠝ ⠆⠨⠹ ⠨⠅ ⠂

$\sin 2\ \theta = 1/2$

⠎⠊⠝ ⠆⠨⠹ ⠨⠅ ⠂⠌⠆

$2\ \theta = 30°,$

⠆⠨⠹ ⠨⠅ ⠒⠴⠙⠛⠠⠂

$2\ \theta = 150°,$

⠆⠨⠹ ⠨⠅ ⠂⠢⠴⠙⠛⠠⠂

$2\ \theta = 30° + 360° = 390°,$

⠆⠨⠹ ⠨⠅ ⠒⠴⠙⠛ ⠬⠒⠖⠴⠙⠛ ⠨⠅ ⠒⠔⠴⠙⠛⠠⠂

$2\ \theta = 150° + 360° = 510°$

⠆⠨⠹ ⠨⠅ ⠂⠢⠴⠙⠛ ⠬⠒⠖⠴⠙⠛ ⠨⠅ ⠢⠂⠴⠙⠛

$\therefore\ \theta = 15°,$

⠹⠋ ⠨⠹ ⠨⠅ ⠂⠢⠙⠛⠠⠂

$\theta = 75°,$

⠨⠹ ⠨⠅ ⠶⠢⠙⠛⠠⠂

$\theta = 195°,$

⠨⠹ ⠨⠅ ⠂⠔⠢⠙⠛⠠⠂

$\theta = 255°.$

⠨⠹ ⠨⠅ ⠆⠢⠢⠙⠛⠠⠲

XXVI

A PROBLEM IN ANALYTIC GEOMETRY

132. A Locus Problem:

Find the equation of the locus of a point which moves so that its distance from the line $3x + 4y = 10$

⠒⠭⠬⠲⠽ ⠨⠅ ⠂⠴ is always [99] equal to its distance from the point (2, 3).
⠪⠆⠠ ⠒⠕

Solution:

Let X and Y be the coordinates of a point in the required locus. Let d_l ⠙⠰⠇ denote the distance of the point (X, Y) ⠪⠠⠭⠠ ⠠⠽⠕ from the given line, and let d_p ⠙⠰⠏ denote the distance of (X, Y) ⠪⠠⠭⠠ ⠠⠽⠕ from the given point. Then it is required that $d_l = d_p$ ⠙⠰⠇ ⠨⠅ ⠙⠰⠏.

$$d_l = \frac{3X + 4Y - 10}{\sqrt{3^2 + 4^2}} = \frac{3X + 4Y - 10}{5}$$

⠙⠰⠇ ⠨⠅ ⠹⠒⠠⠭⠬⠲⠠⠽⠤⠂⠴⠌⠜⠒⠣⠬⠲⠣⠻⠼
⠨⠅ ⠹⠒⠠⠭⠬⠲⠠⠽⠤⠂⠴⠌⠢⠼

$$d_p = \sqrt{(X - 2)^2 + (Y - 3)^2} = \sqrt{X^2 - 4X + 4 + Y^2 - 6Y + 9}$$

⠙⠰⠏ ⠨⠅ ⠜⠪⠠⠭⠤⠆⠕⠣⠬⠪⠠⠽⠤⠒⠕⠣⠻
⠨⠅ ⠜⠠⠭⠣⠤⠲⠠⠭⠬⠲⠬⠠⠽⠣⠤⠖⠠⠽⠬⠔⠻

$$\frac{3X + 4Y - 10}{5} = \sqrt{X^2 + Y^2 - 4X - 6Y + 13}$$

⠹⠒⠠⠭⠬⠲⠠⠽⠤⠂⠴⠌⠢⠼
⠨⠅ ⠜⠠⠭⠣⠬⠠⠽⠣⠤⠲⠠⠭⠤⠖⠠⠽⠬⠂⠒⠻

$$3X + 4Y - 10 = 5\sqrt{X^2 + Y^2 - 4X - 6Y + 13}$$

⠒⠠⠭⠬⠲⠠⠽⠤⠂⠴
⠨⠅ ⠢⠜⠠⠭⠣⠬⠠⠽⠣⠤⠲⠠⠭⠤⠖⠠⠽⠬⠂⠒⠻

$$(3X + 4Y - 10)^2 = 25(X^2 + Y^2 - 4X - 6Y + 13)$$

⠪⠒⠠⠭⠬⠲⠠⠽⠤⠂⠴⠕⠣
⠨⠅ ⠆⠢⠪⠠⠭⠣⠬⠠⠽⠣⠤⠲⠠⠭⠤⠖⠠⠽⠬⠂⠒⠕

$9X^2 + 24XY + 16Y^2 - 60X - 80Y + 100 = 25X^2 + 25Y^2 - 100X - 150Y + 325$

⠔⠠⠭⠣⠬⠆⠲⠠⠭⠠⠽⠬⠂⠖⠠⠽⠣⠤⠖⠴⠠⠭⠤⠦⠴⠠⠽
⠬⠂⠴⠴ ⠨⠅ ⠆⠢⠠⠭⠣⠬⠆⠢⠠⠽⠣⠤⠂⠴⠴⠠⠭
⠤⠂⠢⠴⠠⠽⠬⠒⠆⠢

$-16X^2 + 24XY - 9Y^2 + 40X + 70Y - 225 = 0$

⠤⠂⠖⠠⠭⠣⠬⠆⠲⠠⠭⠠⠽⠤⠔⠠⠽⠣⠬⠲⠴⠠⠭⠬⠶⠴⠠⠽
⠤⠆⠆⠢ ⠨⠅ ⠴

$16X^2 - 24XY + 9Y^2 - 40X - 70Y + 225 = 0$

⠂⠖⠠⠭⠣⠤⠆⠲⠠⠭⠠⠽⠬⠔⠠⠽⠣⠤⠲⠴⠠⠭⠤⠶⠴⠠⠽
⠬⠆⠆⠢ ⠨⠅ ⠴ [100]

XXVII

PROBLEMS FROM THE CALCULUS

133. Differentiation:

Find the derivative of $y = \frac{\sqrt{5 - 2x}}{2x + 1}$ ⠽ ⠨⠅ ⠹⠜⠢⠤⠆⠭⠻⠌⠆⠭⠬⠂⠼

Solution:

$$y = \frac{\sqrt{5 - 2x}}{2x + 1}$$

⠽ ⠨⠅ ⠹⠜⠢⠤⠆⠭⠻⠌⠆⠭⠬⠂⠼

$$= \frac{(5 - 2x)^{1/2}}{2x + 1}$$

⠨⠅ ⠹⠪⠢⠤⠆⠭⠕⠘⠂⠌⠆⠐⠌⠆⠭⠬⠂⠼

$$\frac{dy}{dx} = \frac{(2x + 1)(1/2)(5 - 2x)^{-1/2}(-2) - (5 - 2x)^{1/2}(2)}{(2x + 1)^2}$$

⠙⠽⠌⠙⠭ ⠨⠅ ⠹⠪⠆⠭⠬⠂⠕⠪⠂⠌⠆⠕
⠡⠪⠢⠤⠆⠭⠕⠘⠤⠂⠌⠆⠐⠪⠤⠆⠕
⠤⠪⠢⠤⠆⠭⠕⠘⠂⠌⠆⠐⠪⠆⠕⠌⠪⠆⠭⠬⠂⠕⠣⠼

$$= \frac{-(2x + 1)(5 - 2x)^{-1/2} - 2(5 - 2x)^{1/2}}{(2x + 1)^2}$$

⠨⠅ ⠹⠤⠪⠆⠭⠬⠂⠕⠪⠢⠤⠆⠭⠕⠘⠤⠂⠌⠆
⠐⠤⠆⠪⠢⠤⠆⠭⠕⠘⠂⠌⠆⠐⠌⠪⠆⠭⠬⠂⠕⠣⠼

$$= \frac{-(5 - 2x)^{-1/2}}{2x + 1} - \frac{2(5 - 2x)^{1/2}}{(2x + 1)^2}$$

⠨⠅ ⠹⠤⠪⠢⠤⠆⠭⠕⠘⠤⠂⠌⠆⠐⠌⠆⠭⠬⠂⠼
⠤⠹⠆⠪⠢⠤⠆⠭⠕⠘⠂⠌⠆⠐⠌⠪⠆⠭⠬⠂⠕⠣⠼

$$= \frac{-1}{(2x + 1)\sqrt{5 - 2x}} - \frac{2\sqrt{5 - 2x}}{(2x + 1)^2}$$

⠨⠅ ⠹⠤⠂⠌⠪⠆⠭⠬⠂⠕⠜⠢⠤⠆⠭⠻⠼
⠤⠹⠆⠜⠢⠤⠆⠭⠻⠌⠪⠆⠭⠬⠂⠕⠣⠼

134. Radius of Curvature:

Find an expression for the radius of curvature of the catenary

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

⠽ ⠨⠅ ⠁⠌⠆⠼⠪⠑⠘⠭⠌⠁⠐⠬⠑⠘⠤⠭⠌⠁⠐⠕

Solution:

$$R = \frac{(1 + y'^2)^{3/2}}{y''}$$

⠠⠗ ⠨⠅ ⠹⠪⠂⠬⠽⠄⠣⠕⠘⠒⠌⠆⠐⠌⠽⠄⠄⠼

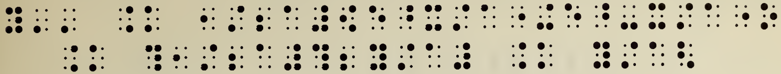
$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

⠽ ⠨⠅ ⠁⠌⠆⠼⠪⠑⠘⠭⠌⠁⠐⠬⠑⠘⠤⠭⠌⠁⠐⠕

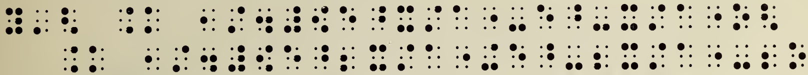
$$y' = \frac{1}{2}\left(e^{x/a} - e^{-x/a}\right)$$

⠽⠄ ⠨⠅ ⠂⠌⠆⠼⠪⠑⠘⠭⠌⠁⠐⠤⠑⠘⠤⠭⠌⠁⠐⠕

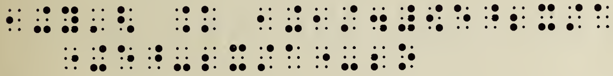
$$y'' = \frac{1}{2a}\left(e^{x/a} + e^{-x/a}\right) = \frac{1}{2a}\,\frac{2y}{a} = \frac{y}{a^2} \quad [101]$$



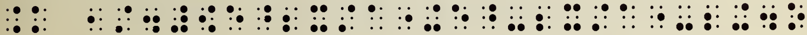
$$y'^2 = \frac{1}{4}\left(e^{x/a} - e^{-x/a}\right)^2 = \frac{1}{4}\left(e^{2x/a} + e^{-2x/a} - 2\right)$$



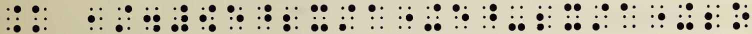
$$1 + y'^2 = 1 + \frac{1}{4}\left(e^{2x/a} + e^{-2x/a} - 2\right)$$



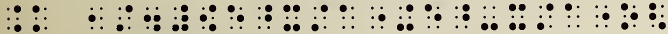
$$= \frac{1}{4}\left(e^{2x/a} + e^{-2x/a} - 2 + 4\right)$$



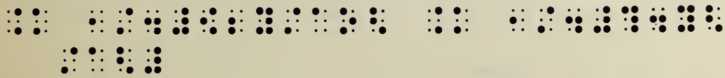
$$= \frac{1}{4}\left(e^{2x/a} + e^{-2x/a} + 2\right)$$



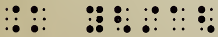
$$= \frac{1}{4}\left(e^{x/a} + e^{-x/a}\right)^2$$



$$= \frac{1}{4}\left(\frac{2y}{a}\right)^2 = \frac{1}{4}\,\frac{4y^2}{a^2}$$



$$= \frac{y^2}{a^2}$$



$$R = \frac{\left(\frac{y^2}{a^2}\right)^{3/2}}{\frac{y}{a^2}}$$

⠠⠗ ⠨⠅ ⠠⠹⠪⠽⠣⠌⠁⠣⠕⠘⠒⠌⠆⠐⠌⠌⠽⠌⠁⠣⠠⠼

$$= \frac{\frac{(y^2)^{3/2}}{(a^2)^{3/2}}}{\frac{y}{a^2}}$$

⠨⠅ ⠠⠹⠽⠣⠘⠒⠌⠆⠐⠌⠁⠣⠘⠒⠌⠆⠐⠌⠌⠽⠌⠁⠣⠠⠼

$$= \frac{\frac{y^3}{a^3}}{\frac{y}{a^2}}$$ ⠨⠅ ⠹⠽⠩⠌⠁⠩⠼⠌⠌⠹⠽⠌⠁⠣⠼

$$= \frac{y^3}{a^3}\,\frac{a^2}{y} = \frac{a^2y^3}{a^3y}$$

⠨⠅ ⠹⠽⠩⠌⠁⠩⠼⠹⠁⠣⠌⠽⠼ ⠨⠅ ⠁⠣⠽⠩⠌⠁⠩⠽

$$= \frac{y^2}{a}$$ ⠨⠅ ⠽⠣⠌⠁

135. A Differential Equation:

Find the general solution of the differential equation $(xe^{y/x} + y)dx - xdy = 0$

⠪⠭⠑⠘⠽⠌⠭⠐⠬⠽⠕⠙⠭⠤⠭⠙⠽ ⠨⠅ ⠴

Solution:

$(xe^{y/x} + y)dx - xdy = 0$

⠪⠭⠑⠘⠽⠌⠭⠐⠬⠽⠕⠙⠭⠤⠭⠙⠽ ⠨⠅ ⠴

Let $y = vx$. ⠽ ⠨⠅ ⠧⠭

Then $dy = vdx + xdv$ ⠙⠽ ⠨⠅ ⠧⠙⠭⠬⠭⠙⠧

and $y/x = v$ ⠽⠌⠭ ⠨⠅ ⠧

Substituting, we have $(xe^v + vx)dx - x(vdx + xdv) = 0$

⠪⠭⠑⠘⠧⠐⠬⠧⠭⠕⠙⠭⠤⠭⠪⠧⠙⠭⠬⠭⠙⠧⠕ ⠨⠅ ⠴

$(e^v + v)dx - (vdx + xdv) = 0$

⠪⠑⠘⠧⠐⠬⠧⠕⠙⠭⠤⠪⠧⠙⠭⠬⠭⠙⠧⠕ ⠨⠅ ⠴

$e^v dx + vdx - vdx - xdv = 0$

⠑⠘⠧⠐⠙⠭⠬⠧⠙⠭⠤⠧⠙⠭⠤⠭⠙⠧ ⠨⠅ ⠴

$e^v dx - xdv = 0$

⠑⠘⠧⠐⠙⠭⠤⠭⠙⠧ ⠨⠅ ⠴ [102]

$\frac{dx}{x} - \frac{dv}{e^v} = 0$

⠹⠙⠭⠌⠭⠼⠤⠹⠙⠧⠌⠑⠘⠧⠐⠼ ⠨⠅ ⠴

$\log x + e^{-v} = k$

⠇⠕⠛ ⠭ ⠬⠑⠘⠤⠧ ⠨⠅ ⠅

Since $v = y/x$,

⠧ ⠨⠅ ⠽⠌⠭

we have $\log x + e^{-y/x} = k$

⠇⠕⠛ ⠭ ⠬⠑⠘⠤⠽⠌⠭ ⠨⠅ ⠅

136. A Definite Integral:

Find the value of $\int_{-1}^{2} (3x^2 + 2)dx$.

⠮⠰⠤⠂⠘⠆⠐⠪⠒⠭⠣⠬⠆⠕⠙⠭

Solution:

$\int_{-1}^{2} (3x^2 + 2)dx$

⠮⠰⠤⠂⠘⠆⠐⠪⠒⠭⠣⠬⠆⠕⠙⠭

$= \int_{-1}^{2} 3x^2dx + \int_{-1}^{2} 2dx$

⠨⠅ ⠮⠰⠤⠂⠘⠆⠐⠒⠭⠣⠙⠭⠬⠮⠰⠤⠂⠘⠆⠐⠆⠙⠭

$= [x^3]_{-1}^{2} + [2x]_{-1}^{2}$

⠨⠅ ⠷⠭⠩⠾⠰⠤⠂⠘⠆⠐⠬⠷⠆⠭⠾⠰⠤⠂⠘⠆

$=[(2^3) - (-1)^3] + [(2 \times 2) - (2 \times -1)]$

⠨⠅ ⠷⠪⠆⠩⠕⠤⠪⠤⠂⠕⠩⠾⠬⠷⠪⠆⠡⠆⠕⠤⠪⠆⠡⠤⠂⠕⠾

$= (8 + 1) + (4 + 2) = 9 + 6 = 15.$

⠨⠅ ⠪⠦⠬⠂⠕⠬⠪⠲⠬⠆⠕ ⠨⠅ ⠔⠬⠖ ⠨⠅ ⠂⠢⠠⠲

137. A Double Integral:

Find the volume which is bounded by the xy-plane and by the elliptic paraboloid $4z = 16 - 4x^2 - y^2$.

⠲⠵ ⠨⠅ ⠂⠖⠤⠲⠭⠣⠤⠽⠣

Solution:

$4z = 16 - 4x^2 - y^2$ ⠲⠵ ⠨⠅ ⠂⠖⠤⠲⠭⠣⠤⠽⠣

$z = 4 - x^2 - y^2/4$ ⠵ ⠨⠅ ⠲⠤⠭⠣⠤⠽⠣⠌⠲

To find the limits of integration, let $z = 0$ ⠵ ⠨⠅ ⠴⠠⠲. Then

$4x^2 + y^2 = 16$ ⠲⠭⠣⠬⠽⠣ ⠨⠅ ⠂⠖

$y^2 = 16 - 4x^2$ ⠽⠣ ⠨⠅ ⠂⠖⠤⠲⠭⠣

$y = \sqrt{16 - 4x^2}$ ⠽ ⠨⠅ ⠜⠂⠖⠤⠲⠭⠣⠻

$y = 2\sqrt{4 - x^2}$ ⠽ ⠨⠅ ⠆⠜⠲⠤⠭⠣⠻

and when y is also equal to 0, $\sqrt{16 - 4x^2} = 0$ ⠜⠂⠖⠤⠲⠭⠣⠻ ⠨⠅ ⠴

$16 - 4x^2 = 0$ ⠂⠖⠤⠲⠭⠣ ⠨⠅ ⠴

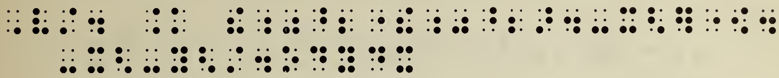
$4x^2 = 16$ ⠲⠭⠣ ⠨⠅ ⠂⠖

$x^2 = 4$ ⠭⠣ ⠨⠅ ⠲

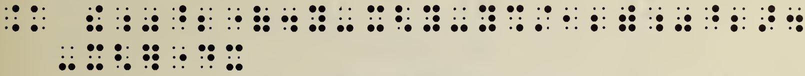
$x = \pm 2$ ⠭ ⠨⠅ ⠬⠤⠆

We consider the volume in the first **[103]** quadrant. We then have

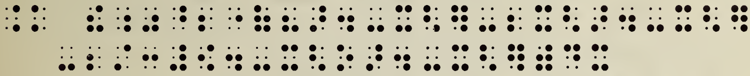
$$V/4 = \int_0^2 \int_0^{2\sqrt{4-x^2}} (4 - x^2 - y^2/4)dydx$$



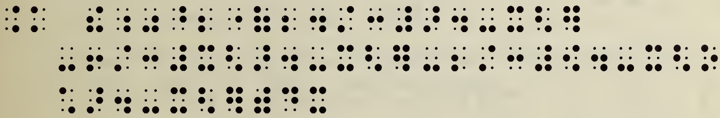
$$= \int_0^2 [4y - x^2y - y^3/12]_0^{2\sqrt{4-x^2}} dx$$



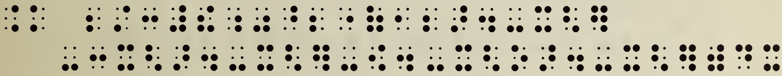
$$= \int_0^2 [8\sqrt{4 - x^2} - 2x^2\sqrt{4 - x^2} - 2/3(4 - x^2)\sqrt{4 - x^2}]dx$$



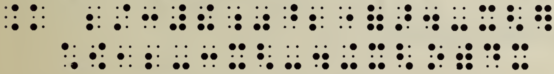
$$= \int_0^2 [24/3\sqrt{4 - x^2} - 6/3(x^2)\sqrt{4 - x^2} - 2/3(4 - x^2)\sqrt{4 - x^2}]dx$$



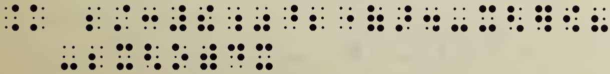
$$= 2/3 \int_0^2 [12\sqrt{4 - x^2} - 3x^2\sqrt{4 - x^2} - (4 - x^2)\sqrt{4 - x^2}]dx$$



$$= 2/3 \int_0^2 [\sqrt{4 - x^2}(12 - 3x^2 - 4 + x^2)]dx$$



$$= 2/3 \int_0^2 [\sqrt{4 - x^2}(8 - 2x^2)]dx$$



$= 4/3 \int_0^2 [\sqrt{4 - x^2}(4 - x^2)]dx$

⠨⠅ ⠲⠌⠒⠼⠮⠰⠴⠘⠆⠐⠷⠜⠲⠤⠭⠣⠻⠪⠲⠤⠭⠣⠕⠾⠙⠭

$= 4/3 \int_0^2 (4 - x^2)^{3/2}\ dx$

⠨⠅ ⠲⠌⠒⠼⠮⠰⠴⠘⠆⠐⠪⠲⠤⠭⠣⠕⠘⠒⠌⠆⠐⠙⠭

Let $x = 2 \sin v$ ⠭ ⠨⠅ ⠆⠎⠊⠝ ⠧. Then

$x^2 = 4 \sin^2 v$ ⠭⠣ ⠨⠅ ⠲⠎⠊⠝⠣ ⠧

$4 - x^2 = 4 - 4 \sin^2 v = 4(1 - \sin^2 v) = 4 \cos^2 v$

⠲⠤⠭⠣ ⠨⠅ ⠲⠤⠲⠎⠊⠝⠣ ⠧
⠨⠅ ⠲⠪⠂⠤⠎⠊⠝⠣ ⠧⠕ ⠨⠅ ⠲⠉⠕⠎⠣ ⠧

$(4 - x^2)^{3/2} = 8 \cos^3 v$

⠪⠲⠤⠭⠣⠕⠘⠒⠌⠆ ⠨⠅ ⠦⠉⠕⠎⠩ ⠧

$dx = 2 \cos v\ dv.$ ⠙⠭ ⠨⠅ ⠆⠉⠕⠎ ⠧ ⠙⠧⠠⠲

Making the proper substitutions, we have [104] $V/4 = 4/3 \int_{x=0}^{2} 8 \cos^3 v \times 2 \cos v\ dv$

⠠⠧⠌⠲ ⠨⠅ ⠲⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠦⠉⠕⠎⠩ ⠧
⠡⠆⠉⠕⠎ ⠧ ⠙⠧

$= 4/3 \int_{x=0}^{2} 16 \cos^4 v\ dv$

⠨⠅ ⠲⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠂⠖⠉⠕⠎⠘⠲ ⠧ ⠙⠧

$= 64/3 \int_{x=0}^{2} \cos^4 v\ dv$

⠨⠅ ⠖⠲⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠉⠕⠎⠘⠲ ⠧ ⠙⠧

We consider the following trigonometric relations:

$\cos^4 v = \cos^2 v \times \cos^2 v$

⠉⠕⠎⠘⠲ ⠧ ⠨⠅ ⠉⠕⠎⠣ ⠧ ⠡⠉⠕⠎⠣ ⠧

$\cos 2v = 2 \cos^2 v - 1$

⠉⠕⠎ ⠆⠧ ⠨⠅ ⠆⠉⠕⠎⠣ ⠧ ⠤⠂

$2 \cos^2 v = \cos 2v + 1$

⠆⠉⠕⠎⠣ ⠧ ⠨⠅ ⠉⠕⠎ ⠆⠧ ⠬⠂

$\cos^2 v = \frac{1}{2}(\cos 2v + 1)\cos^4 v$

⠉⠕⠎⠣ ⠧ ⠨⠅ ⠂⠌⠆⠼⠪⠉⠕⠎ ⠆⠧ ⠬⠂⠕
⠉⠕⠎⠘⠲ ⠧

$= \frac{1}{4}(\cos^2 2v + 2 \cos 2\ v + 1)$

⠨⠅ ⠂⠌⠲⠼⠪⠉⠕⠎⠣ ⠆⠧
⠬⠆⠉⠕⠎ ⠆⠧ ⠬⠂⠕

Making this substitution, we have

$V/4 = 64/3 \int_{x=0}^{2} \cos^4 v \, dv$

⠠⠧⠌⠲ ⠨⠅ ⠖⠲⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠉⠕⠎⠘⠲ ⠧
⠙⠧

$= 64/3 \int_{x=0}^{2} [\frac{1}{4}(\cos^2 2v + 2 \cos 2v + 1)] dv$

⠨⠅ ⠖⠲⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠷⠂⠌⠲⠼⠪⠉⠕⠎⠣ ⠆⠧
⠬⠆⠉⠕⠎ ⠆⠧ ⠬⠂⠕⠾⠙⠧

$= 16/3 \int_{x=0}^{2} [\cos^2 2v + 2 \cos 2v + 1] dv$

⠨⠅ ⠂⠖⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠷⠉⠕⠎⠣ ⠆⠧
⠬⠆⠉⠕⠎ ⠆⠧ ⠬⠂⠾⠙⠧

Returning to trigonometric considerations:

$\cos 4v = 2 \cos^2 2v - 1$

⠉⠕⠎ ⠲⠧ ⠨⠅ ⠆⠉⠕⠎⠣ ⠆⠧ ⠤⠂

$2 \cos^2 2v = \cos 4v + 1$

⠆⠉⠕⠎⠣ ⠆⠧ ⠨⠅ ⠉⠕⠎ ⠲⠧ ⠬⠂

$\cos^2 2v = \frac{1}{2}(\cos 4v + 1)$

⠉⠕⠎⠣ ⠆⠧ ⠨⠅ ⠂⠌⠆⠼⠪⠉⠕⠎ ⠲⠧ ⠬⠂⠕

Substituting again, we obtain

$$V/4 = 16/3 \int_{x=0}^{2} [\tfrac{1}{2}(\cos 4v + 1) + 2 \cos 2v + 1]dv$$

⠠⠧⠌⠲
⠨⠅ ⠂⠖⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆
⠐⠷⠂⠌⠆⠼⠪⠉⠕⠎ ⠲⠧ ⠬⠂⠕
⠬⠆⠉⠕⠎ ⠆⠧ ⠬⠂⠾⠙⠧

$$= 16/3 \int_{x=0}^{2} [\tfrac{1}{2}(\cos 4v + 1\ [105] + 4 \cos 2v + 2)]\ dv$$

⠨⠅ ⠂⠖⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠷⠂⠌⠆⠼⠪⠉⠕⠎ ⠲⠧
⠬⠂⠬⠲⠉⠕⠎ ⠆⠧ ⠬⠆⠕⠾⠙⠧

$$= 8/3 \int_{x=0}^{2} (\cos 4v + 4 \cos 2v + 3)dv$$

⠨⠅ ⠦⠌⠒⠼⠮⠰⠭⠨⠅⠴⠘⠆⠐⠪⠉⠕⠎ ⠲⠧
⠬⠲⠉⠕⠎ ⠆⠧ ⠬⠒⠕⠙⠧

$$= 8/3 \left[\tfrac{1}{4} \sin 4v + 2 \sin 2v + 3v\right]_{x=0}^{2}$$

⠨⠅ ⠦⠌⠒⠼⠷⠂⠌⠲⠼⠎⠊⠝ ⠲⠧ ⠬⠆⠎⠊⠝ ⠆⠧
⠬⠒⠧⠾⠰⠭⠨⠅⠴⠘⠆

$$= 8/3 \left[\tfrac{1}{4} \sin 4v + 8/4 \sin 2v + 12/4(v)\right]_{x=0}^{2}$$

⠨⠅ ⠦⠌⠒⠼⠷⠂⠌⠲⠼⠎⠊⠝ ⠲⠧ ⠬⠦⠌⠲⠼⠎⠊⠝ ⠆⠧
⠬⠂⠆⠌⠲⠼⠧⠾⠰⠭⠨⠅⠴⠘⠆

$$= 2/3 \left[\sin 4v + 8 \sin 2v + 12v\right]_{x=0}^{2}$$

⠨⠅ ⠆⠌⠒⠼⠷⠎⠊⠝ ⠲⠧ ⠬⠦⠎⠊⠝ ⠆⠧ ⠬⠂⠆⠧⠾
⠰⠭⠨⠅⠴⠘⠆

Since $x = 2 \sin v$ ⠭ ⠨⠅ ⠆⠎⠊⠝ ⠧⠠⠂ , $\sin v = x/2$ ⠎⠊⠝ ⠧
⠨⠅ ⠭⠌⠆⠠⠂ , $v = \text{arc} \sin x/2$ ⠧ ⠨⠅ ⠁⠗⠉⠎⠊⠝ ⠭⠌⠆ .

Making this substitution, we obtain

$$V/4 = 2/\,3\, \left[\sin\,(4\ \text{arc}\sin x/2) + 8 \sin\,(2\ \text{arc}\sin x/2) + 12\ \text{arc}\sin x/2\right]_0^2$$

⠠⠧⠌⠲ ⠨⠅ ⠆⠌⠒⠼⠷⠎⠊⠝ ⠪⠲⠁⠗⠉⠎⠊⠝ ⠭⠌⠆⠕
⠬⠦⠎⠊⠝ ⠪⠆⠁⠗⠉⠎⠊⠝ ⠭⠌⠆⠕
⠬⠂⠆⠁⠗⠉⠎⠊⠝ ⠭⠌⠆⠾⠰⠴⠘⠆

$$= 2/3\ [\sin 2\pi - \sin 0] + 2/3\ [8 \sin \pi - 8 \sin 0] + 2/3\ [6\pi - 0]$$

⠨⠅ ⠆⠌⠒⠼⠷⠎⠊⠝ ⠆⠨⠏ ⠤⠎⠊⠝ ⠴⠾
⠬⠆⠌⠒⠼⠷⠦⠎⠊⠝ ⠨⠏ ⠤⠦⠎⠊⠝ ⠴⠾
⠬⠆⠌⠒⠼⠷⠖⠨⠏⠤⠴⠾

$$= 2/3\ [0 - 0] + 2/3\ [0 - 0] + 2/3\ [6\pi]$$

⠨⠅ ⠆⠌⠒⠼⠷⠴⠤⠴⠾⠬⠆⠌⠒⠼⠷⠴⠤⠴⠾
⠬⠆⠌⠒⠼⠷⠖⠨⠏⠾

$$= 2/3\ [6\pi] = 4\pi$$

⠨⠅ ⠆⠌⠒⠼⠷⠖⠨⠏⠾ ⠨⠅ ⠲⠨⠏

$$V = 4 \times 4\pi = 16\pi$$

⠠⠧ ⠨⠅ ⠲⠡⠲⠨⠏ ⠨⠅ ⠂⠖⠨⠏ [106]

LIST OF MATHEMATICAL SYMBOLS

Note: The following is a list of symbols used in the Nemeth Code of Braille Mathematics Notation. In listing the symbols, the standard arrangement of the sixty-three Braille characters has been followed, and it is in accordance with this arrangement that the mathematical symbols have been "alphabetized." For convenience, the standard arrangement of the sixty-three Braille characters on seven lines is given below:

Line 1.	a ⠁	b ⠃	c ⠉	d ⠙	e ⠑	f ⠋	g ⠛	h ⠓	i ⠊	j ⠚
Line 2.	k ⠅	l ⠇	m ⠍	n ⠝	o ⠕	p ⠏	q ⠟	r ⠗	s ⠎	t ⠞
Line 3.	u ⠥	v ⠧	x ⠭	y ⠽	z ⠵	and ⠯	for ⠿	of ⠷	the ⠮	with ⠾
Line 4.	ch ⠡	gh ⠣	sh ⠩	th ⠹	wh ⠱	ed ⠫	er ⠻	ou ⠳	ow ⠪	w ⠺
Line 5.	ea , ⠂	be, bb ; ⠆	con, cc : ⠒	dis, dd . ⠲	en ⠢	ff ! ⠖	gg ( ) ⠶	" ? ⠦	in ⠔	" ⠴
Line 6.	st Fraction-line ⠌	ing ⠬	ble Numeral sign ⠼	ar Poetry sign ⠜	' Apostrophe ⠄	com Hyphen ⠤				
Line 7.	Accent sign ⠈	Italic, Decimal Sign ⠨	Letter Sign ⠰	Capital Sign ⠠						

Used in Forming Contractions

⠘ ⠸ ⠐ [107]

⠁ small Roman **a** — § 37, p. 18, (Br. p. 38).

⠁⠗⠉ abbreviation for "arc" (of a circle or curve) — § 96, p. 30, (Br. p. 68); used in the representation of the inverse trigonometric and hyperbolic functions, § 112, p. 36, (Br. p. 80); § 112, p. 36, (Br. p. 81).

⠃ small Roman **b** — § 37, p. 18, (Br. p. 38).

⠉ small Roman **c** — § 37, p. 18, (Br. p. 38).

⠉⠼ "cent" sign — § 10, p. 4, (Br. p. 9).

⠉⠈⠇ abbreviation for "circle" — § 96, p. 31, (Br. p. 68).

⠙ small Roman **d** — § 37, p. 18, (Br. p. 38).

⠙⠛ abbreviation for "degrees" — § 14, p. 6, (Br. p. 14).

⠑ small Roman **e** — § 37, p. 18, (Br. p. 38).

⠋ small Roman **f** — § 37, p. 18, (Br. p. 38).

⠋⠞ abbreviation for "feet" — § 14, p. 7, (Br. p. 14).

⠛ small Roman **g** — § 37, p. 18, (Br. p. 38).

⠓ small Roman **h** — § 37, p. 18, (Br. p. 38).

⠊ small Roman **i** — § 37, p. 18, (Br. p. 38).

⠊⠝ abbreviation for "inches" — § 14, p. 7, (Br. p. 14).

⠚ small Roman **j** — § 37, p. 18, (Br. p. 38).

⠅ small Roman **k** — § 37, p. 18, (Br. p. 38).

⠇ small Roman **l** — § 37, p. 18, (Br. p. 38).

⠍ small Roman **m** — § 37, p. 18, (Br. p. 38).

⠍⠊⠝ abbreviation for "minutes" — § 14, p. 7, (Br. p. 15). [108]

⠍⠝ abbreviation for "minutes" — § 14, p. 7, (Br. p. 15).

⠝ small Roman **n** — § 37, p. 18, (Br. p. 38).

⠕ small Roman **o** — § 37, p. 18, (Br. p. 38); part of the sign for "divided into" — § 21, p. 11, (Br. p. 23); closing parenthesis — § 49, p. 21, (Br. p. 47).

⠏ small Roman **p** — § 37, p. 18, (Br. p. 38).

⠏⠉ the "per cent" sign — § 12, p. 5, (Br. p. 11).

⠏⠈⠛ abbreviation for "parallelogram" — § 96, p. 31, (Br. p. 69).

⠏⠈⠇ abbreviation for "is parallel to" — § 96, p. 31, (Br. p. 69).

⠏⠈⠏ abbreviation for "is perpendicular to" — § 96, p. 31, (Br. p. 69).

⠟ small Roman **q** — § 37, p. 18, (Br. p. 38).

⠟⠈⠙ abbreviation for "quadrilateral" — § 96, p. 31, (Br. p. 69).

⠗ small Roman **r** — § 37, p. 18, (Br. p. 38).

⠗⠉⠸⠪ abbreviation for "rectangle" — § 96, p. 31, (Br. p. 69).

⠗⠸⠪ abbreviation for "right angle" — § 96, p. 31, (Br. p. 69).

⠎ small Roman **s** — § 37, p. 18, (Br. p. 38).

⠎⠉ abbreviation for "since" — § 94, p. 30, (Br. p. 68).

⠎⠑⠉ abbreviation for "seconds" — § 14, p. 8, (Br. p. 16).

⠎⠈⠟ abbreviation for "square" — § 96, p. 31, (Br. p. 69). [109]

⠞ small Roman **t** — § 37, p. 18, (Br. p. 38).

⠞⠸⠪ abbreviation for "triangle" — § 96, p. 31, (Br. p. 69).

⠥ small Roman **u** — § 37, p. 18, (Br. p. 38).

⠧ small Roman **v** — § 37, p. 18, (Br. p. 38).

⠭ small Roman **x** — § 37, p. 18, (Br. p. 38).

⠽ small Roman **y** — § 37, p. 18, (Br. p. 38).

⠵ small Roman **z** — § 37, p. 18, (Br. p. 38).

⠯ the "factorial" sign — § 102, p. 33, (Br. p. 74).

⠿ the sign for "infinity" — § 79, p. 25, (Br. p. 57).

⠷ opening bracket — § 50, p. 22, (Br. p. 48).

⠮ the integral sign — § 104, p. 34, (Br. p. 75).

⠾ closing bracket — § 50, p. 22, (Br. p. 48).

⠾ ⠷ "is equivalent to" — § 55, p. 23, (Br. p. 51).

⠾ ⠌ ⠷ "is not equivalent to" — § 56, p. 23, (Br. p. 51).

⠡ the multiplication sign — § 18, p. 10, (Br. p. 21); § 120, p. 38, (Br. p. 85).

⠡ ⠡ the multiplication sign represented in ink print by the "dot" — § 47, p. 21, (Br. p. 45).

⠣ "square" (the exponent ^{2}) — § 83, p. 26, (Br. p. 59).

⠩ "cube" (the exponent ^{3}) — § 83, p. 26, (Br. p. 59).

⠹ indicates the beginning of a fraction — § 26, p. 13, (Br. p. 26).

⠹ ⠋ abbreviation for "therefore" — § 94, p. 30, (Br. p. 68).

⠱ the bar — § 105, p. 34, (Br. p. 76). [110]

⠫ the "round **d**" used in the representation of partial derivatives — § 103, p. 34, (Br. p. 75).

⠻ sign used for terminating a radical — § 32, p. 16, (Br. p. 33).

⠳ the sign for "the absolute value of" — § 99, p. 32, (Br. p. 71); used in enclosing determinants — § 100, p. 33, (Br. p. 72).

⠳ ⠳ used in enclosing matrices — § 101, p. 33, (Br. p. 73).

⠪ part of the sign for "divided into" — § 21, p. 11, (Br. p. 23); closing parenthesis — § 49, p. 21, (Br. p. 47).

⠺ small Roman w — § 37, p. 18, (Br. p. 38).

⠂ the digit 1 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠆ the digit 2 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠒ the digit 3 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠒ ⠏ ⠼ the "per cent" sign — § 12, p. 5, (Br. p. 10).

⠒⠒⠕ "approaches" — § 75, p. 24, (Br. p. 54).

⠲ the digit 4 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠲⠼ the dollar sign — § 9, p. 4, (Br. p. 8).

⠢ the digit 5 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠖ the digit 6 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠶ the digit 7 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠦ the digit 8 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠔ the digit 9 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62). **[111]**

⠔⠔ the mathematical asterisk — § 106, p. 34, (Br. p. 77).

⠴ the digit 0 — § 4, p. 3, (Br. p. 6); § 88, p. 28, (Br. p. 62).

⠌ the fraction line — § 19, p. 10, (Br. p. 21).

⠌⠌ the division sign — § 20, p. 10, (Br. p. 21); the main fraction line of a complex fraction — § 29, p. 14, (Br. p. 31).

⠌⠐⠅ "is not less than" — § 65, p. 24, (Br. p. 52).

⠌⠐⠅⠨⠅ "is not less than or equal to" — § 68, p. 24, (Br. p. 53).

⠌⠨⠂ "is not greater than" — § 59, p. 23, (Br. p. 52).

⠌⠨⠂⠨⠅ "is not greater than or equal to" — § 62, p. 23, (Br. p. 52).

⠬ the plus sign — § 16, p. 9, (Br. p. 20).

⠬⠤ "plus or minus" — § 45, p. 20, (Br. p. 44).

⠼ indicates the end of a fraction — § 26, p. 13, (Br. p. 27).

⠜ the radical sign — § 31, p. 15, (Br. p. 32).

⠄ "prime" — § 76, p. 24, (Br. p. 55); used in sequences of abbreviated terms which are related, to separate one submultiple from another — § 15, p. 8, (Br. p. 19).

⠤ the minus sign — § 17, p. 10, (Br. p. 20).

⠤⠬ "minus or plus" — § 46, p. 21, (Br. p. 44).

⠈ level indicator for the second exponential level — § 84, p. 27, (Br. p. 60). [112]

⠈ the "dot" used in the representation of time derivatives — § 108, p. 35, (Br. p. 78).

⠈⠈ level indicator for the third exponential level — § 85, p. 27, (Br. p. 61).

⠘ level indicator for the first exponential level — § 81, p. 26, (Br. p. 58).

⠸ used in the representation of a recurring decimal — § 8, p. 4, (Br. p. 8); indicator for letters of alphabets other than the Roman or Greek — § 41, p. 19, (Br. p. 40); indicator for Roman letters having a form other than the usual form — § 42, p. 19, (Br. p. 41); used in the representation of vectors — § 43, p. 19, (Br. p. 42); used in the representation of the index of a radical — § 33, p. 16, (Br. p. 34).

⠸⠇ "is identically equal to" — § 53, p. 23, (Br. p. 51).

⠸⠿ "varies as" — § 74, p. 24, (Br. p. 53).

⠸⠷ curly opening bracket — § 50, p. 22, (Br. p. 48).

⠸⠾ curly closing bracket — §50, p. 22, (Br. p. 48).

⠸⠹ separates the integral from the fractional part of a mixed number — § 28, p. 13, (Br. p. 29).

⠸⠱ underline — § 107, p. 35, (Br. p. 77).

⠸⠫ "del" — § 109, p. 35, (Br. p. 78).

⠸⠻ used as the radical terminating sign when [113] the radical being terminated is under a larger radical — § 34, p. 16, (Br. p. 35).

⠸⠳ the vinculum — § 51, p. 22, (Br. p. 49); abbreviation for "line segment" — § 96, p. 31, (Br. p. 69).

⠸⠪ abbreviation for "angle" — § 96, p. 31, (Br. p. 69).

⠸⠔⠔ mathematical asterisk — § 106, p. 34, (Br. p. 77).

⠸⠌ the diagonal stroke as distinguished from the horizontal fraction line — § 48, p. 21, (Br. p. 45).

⠸⠌⠇ "is not identical with" — § 54, p. 23, (Br. p. 51).

⠸⠼ used in terminating the fractional part of a mixed number — § 28, p. 13, (Br. p. 29).

⠸⠜ the radical sign when the radical being introduced is under a larger radical — § 34, p. 16, (Br. p. 35).

⠸⠠ indicates the capitalization of a letter from an alphabet other than the Roman or Greek — § 41, p. 19, (Br. p. 40); indicates the capitalization of a Roman letter written in a form other than the usual form — § 42, p. 19, (Br. p. 41).

⠸⠠⠷ large curly opening bracket — § 50, p. 22, (Br. p. 49).

⠸⠠⠾ large curly closing bracket [114] — § 50, p. 22, (Br. p. 49).

⠐ indicator for the base line of writing — § 82, p. 26, (Br. p. 58).

⠐⠅ "is less than" — § 64, p. 24, (Br. p. 52).

⠐⠅⠨⠅ "is less than or equal to" — § 66, p. 24, (Br. p. 52).

⠐⠅⠨⠅⠨⠂ "is less than, equal to, or greater than" — § 73, p. 24, (Br. p. 53).

⠐⠅⠨⠂ "is less than or greater than" — § 71, p. 24, (Br. p. 53).

⠐⠂ "is to" — § 23, p. 12, (Br. p. 24).

⠨ the decimal point — § 7, p. 4, (Br. p. 8); indicator used for small Greek letters — § 39, p. 18, (Br. p. 39).

⠨⠁ small alpha — § 39, p. 18, (Br. p. 39).

⠨⠃ small beta — § 39, p. 18, (Br. p. 39).

⠨⠙ small delta — § 39, p. 18, (Br. p. 39).

⠨⠑ small epsilon — § 39, p. 18, (Br. p. 39).

⠨⠋ small phi — § 39, p. 19, (Br. p. 39).

⠨⠛ small gamma — § 39, p. 18, (Br. p. 39).

⠨⠊ small iota — §39, p. 18, (Br. p. 39).

⠨⠅ small kappa — § 39, p. 18, (Br. p. 39); "is equal to" — § 22, p. 12, (Br. p. 24).

⠨⠅⠐⠅ "is equal to or less than" — § 67, p. 24, (Br. p. 53). [115]

⠨⠅⠨⠂ "is equal to or greater than" — § 61, p. 23, (Br. p. 52).

⠨⠇ small lambda — § 39, p. 18, (Br. p. 39).

⠨⠍ small mu — § 39, p. 18, (Br. p. 39).

⠨⠝ small nu — § 39, p. 18, (Br. p. 39).

⠨⠕ small omicron — § 39, p. 18, (Br. p. 39).

⠨⠏ small pi — § 36, p. 18, (Br. p. 37); § 39, p. 18, (Br. p. 39).

⠨⠗ small rho — § 39, p. 18, (Br. p. 39).

⠨⠎ small sigma — § 39, p. 18, (Br. p. 39).

⠨⠞ small tau — § 39, p. 18, (Br. p. 39).

⠨⠥ small upsilon — § 39, p. 18, (Br. p. 39).

⠨⠭ small xi — § 39, p. 18, (Br. p. 39).

⠨⠽ small psi — § 39, p. 19, (Br. p. 39).

⠨⠵ small zeta — § 39, p. 18, (Br. p. 39).

⠨⠯ small chi — § 39, p. 19, (Br. p. 39).

⠨⠮ alternative form of small sigma — § 39, p. 19, (Br. p. 40).

⠨⠡⠅ "is approximately equal to" — § 57, p. 23, (Br. p. 52).

⠨⠹ small theta — § 39, p. 18, (Br. p. 39).

⠨⠱ small eta — § 39, p. 18, (Br. p. 39).

⠨⠻ terminating radical sign for the smallest of three radicals, the second under the first and the third under the second — § 34, p. 17, (Br. p. 35). [116]

⠨⠺ small omega — § 39, p. 19, (Br. p. 39).

⠨⠂ "is greater than" — § 58, p. 23, (Br. p. 52).

⠨⠂⠐⠅ "is greater than or less than" — § 70, p. 24, (Br. p. 53).

⠨⠂⠨⠅ "is greater than or equal to" — § 60, p. 23, (Br. p. 52).

⠨⠂⠨⠅⠐⠅ "is greater than, equal to, or less than" — § 72, p. 24, (Br. p. 53).

⠨⠌⠅ "is not equal to" — § 52, p. 23, (Br. p. 51).

⠨⠌⠅⠐⠅ "is not equal to or less than" — § 69, p. 24, (Br. p. 53).

⠨⠌⠅⠨⠂ "is not equal to or greater than" — § 63, p. 23, (Br. p. 52).

⠨⠜ radical sign when introducing the smallest of three radicals, the second under the first and the third under the second — § 34, p. 17, (Br. p. 35).

⠨⠠ indicates the capitalization of Greek letters — § 40, p. 19, (Br. p. 40).

⠰ indicator for the first subscript level — § 87, p. 27, (Br. p. 61).

⠰⠆ "as" — § 24, p. 12, (Br. p. 24).

⠰⠰ indicator for the second subscript level — § 89, p. 28, (Br. p. 63).

⠠ indicates the capitalization of Roman [117] letters — § 38, p. 18, (Br. p. 38); § 97, p. 31, (Br. p. 69); precedes a mark of punctuation — § 3, p. 2, (Br. p. 3); the mathematical comma — § 5, p. 4, (Br. p. 7); § 77, p. 25, (Br. p. 56).

⠠⠕ large closing parenthesis — § 49, p. 22, (Br. p. 47).

⠠⠷ large opening bracket — § 50, p. 22, (Br. p. 48).

⠠⠮ large integral sign — § 104, p. 34, (Br. p. 75).

⠠⠾ large closing bracket — § 50, p. 22, (Br. p. 48).

⠠⠹ indicates the beginning of a complex fraction — § 29, p. 14, (Br. p. 30).

⠠⠻ indicates the termination of a large radical sign — § 35, p. 17, (Br. p. 37).

⠠⠳ used in enclosing determinants — § 100, p. 33, (Br. p. 72).

⠠⠪ large opening parenthesis — § 49, p. 22, (Br. p. 47).

⠠⠼ indicates the termination of a complex fraction — § 29, p. 14, (Br. p. 30).

⠠⠜ large radical sign — § 35, p. 17, (Br. p. 37).

⠠⠄ "approaches" — § 75, p. 24, (Br. p. 54).

⠠⠠⠠ mathematical ellipses — § 78, p. 25, (Br. p. 56).

THE END

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